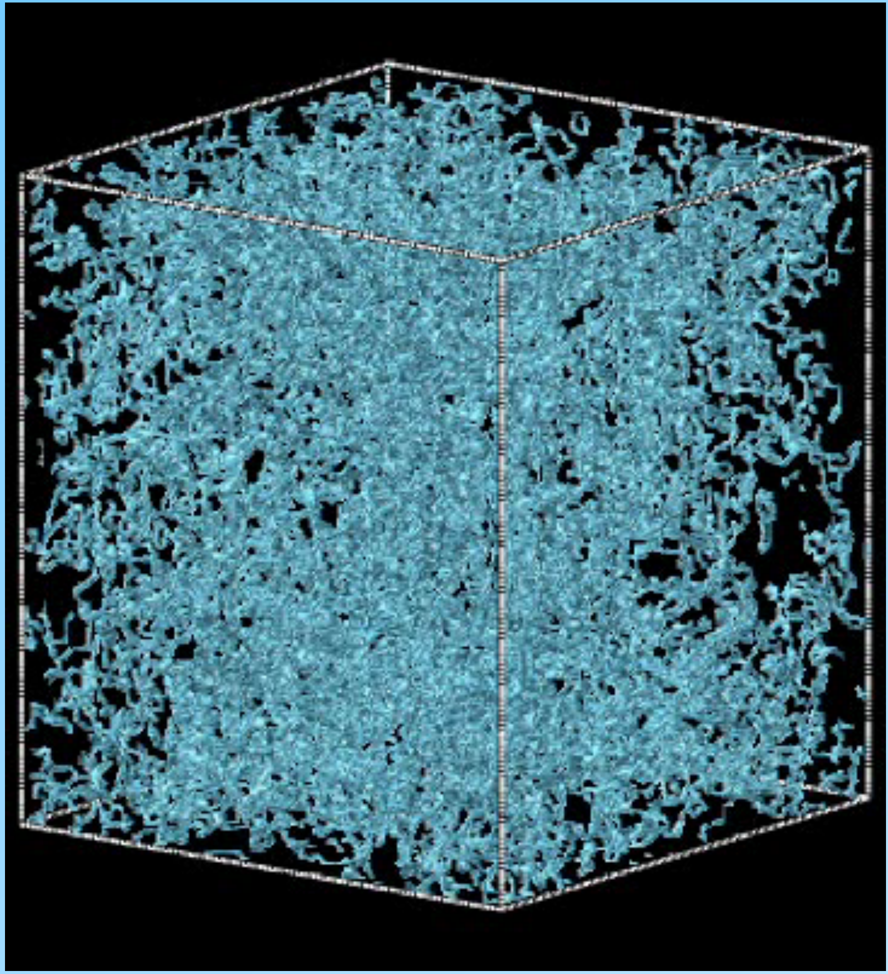
Quantum Hydrodynamics and Turbulence in Atomic Bose-Einstein Condensates



Makoto TSUBOTA
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General review articles on quantum turbulence

- M. Tsubota, K. Kasamatsu and M. Kobayashi, in *Novel Superfluids*, ed. K. H. Bennemann and J. B. Ketterson (Oxford University Press, 2013), Vol.1, p. 156; arXiv:1004.5458
- M. Tsubota, M. Kobayashi and H. Takeuchi, **Physics Reports** 522, 191 (2013); arXiv:1208.0422
- C. F. Barenghi, L. Skrbek, K. R. Sreenivasan, **PNAS** 111 (Suppl. 1), 4647 (2014); arXiv: 1404.1909

1. Introduction

What is quantum turbulence?

Quantum turbulence(QT) basically means

Turbulence in quantum fluids .

The main stages of QT are

- Superfluid helium (since 1950’s)
- Atomic Bose-Einstein condensates(BECs) (since 1995)

Message of my talk

Turbulence is one of the most traditional unresolved problems in physics.

By studying ***quantum turbulence(QT)*** in cold atoms, we can attack the problem by the modern point of view.

Three kinds of QT

(1) QT of quantized vortices

(2) QT of spins

(3) QT of waves

Outline

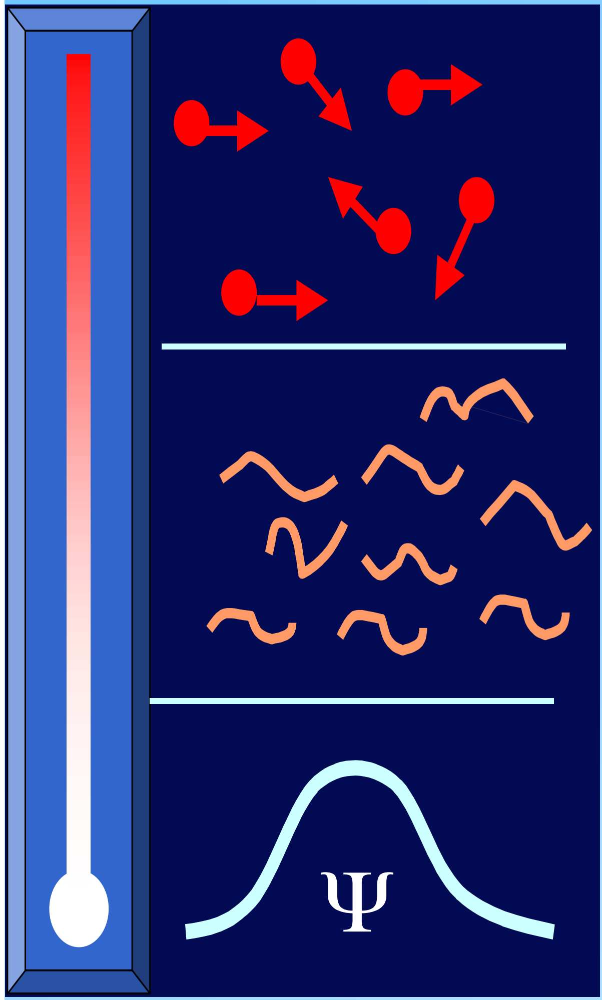
1. Introduction
2. QT in atomic BECs
   2-1. QT in single-component BECs
   2-2. QT in two-component BECs
   2-3. Spin turbulence in spinor BECs
   2-4. Bogoliubov wave turbulence in BECs

{ ***Quantized vortices*** (2-1, 2-2)

Spins (2-3)

Waves (2-4)

Bose-Einstein condensation (BEC) and the macroscopic wave function



Physics of scalar BEC at 0K is described by the macroscopic wave function (order parameter).

$$\Psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\exp\left(i\theta(\mathbf{r},t)\right)$$

$n(\mathbf{r},t)$: Density of the Bose condensate

$\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\nabla\theta(\mathbf{r},t)$: Superfluid velocity

In a weakly interacting BEC, $\Psi(\mathbf{r},t)$ obeys the Gross-Pitaevskii (GP) equation.

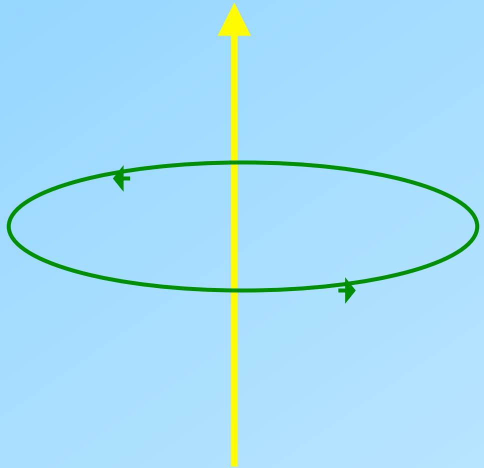
$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t)$$

A quantized vortex is a vortex of superflow in a BEC. Any rotational motion in superfluid is sustained by quantized vortices.

(i) The circulation is quantized.

$$\oint \boldsymbol{v}_s \cdot d\boldsymbol{s} = \kappa n \;\; (n = 0, 1, 2, \cdots)$$
$$\kappa = h/m$$

A vortex with $n \geqq 2$ is unstable.



⇨ **Every vortex has the same circulation.**

(ii) Free from the decay mechanism of the viscous diffusion of the vorticity.

⇨ **The vortex is stable once it is nucleated.**

A quantized vortex is definite and well-defined!!

Models available for simulation of QT

Gross-Pitaevskii (GP) model for the macroscopic wave function

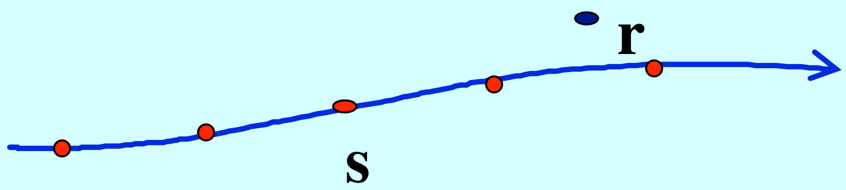
$$\Psi(r) = \sqrt{n_0(r)}e^{i\theta(r)}$$

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ext}}(\mathbf{r}) + g\left|\Psi(\mathbf{r},t)\right|^2\right]\Psi(\mathbf{r},t)$$

ATOMIC BECS

Vortex filament model **Biot-Savart law**

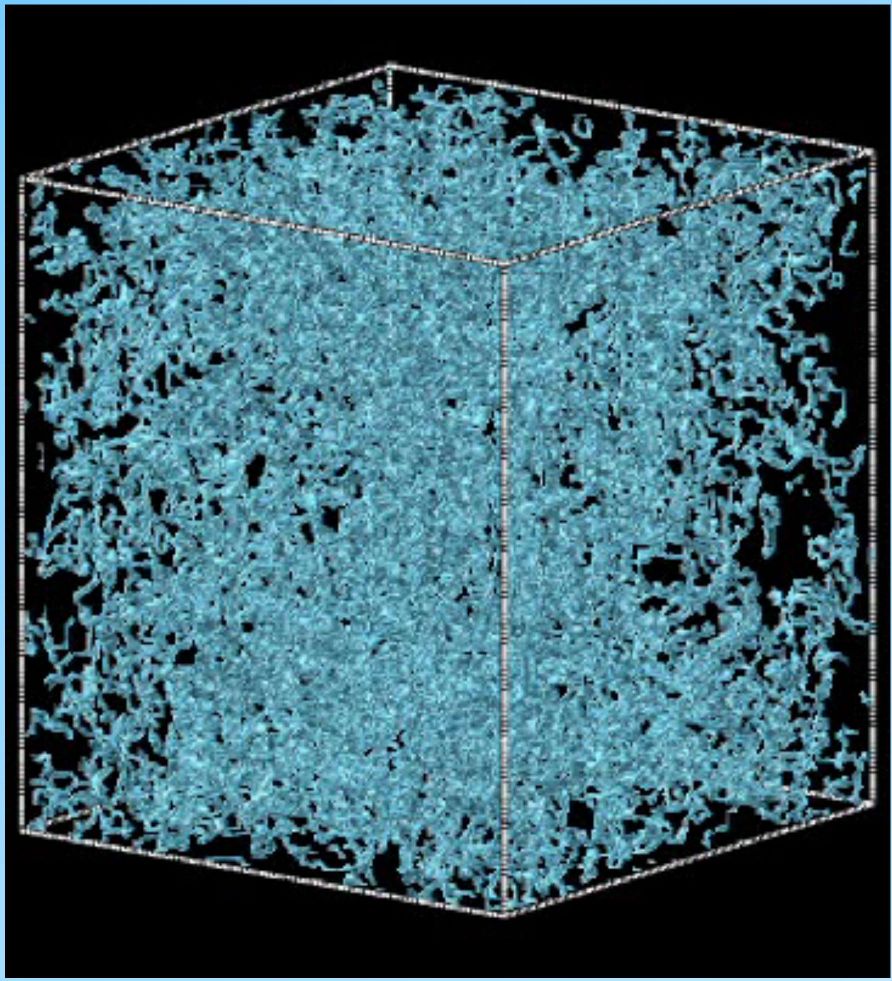
$$\mathbf{v}_s(\mathbf{r}) = \frac{\kappa}{4\pi}\int\frac{(\mathbf{s}-\mathbf{r})\times d\mathbf{s}}{\left|\mathbf{s}-\mathbf{r}\right|^3}$$



A vortex makes the superflow of the Biot-Savart law, and moves with this local flow.

SUPERFLUID HELIUM

Vortex tangle in ***Quantum Turbulence***.



Numerical simulation by the Gross-Pitaevskii model.

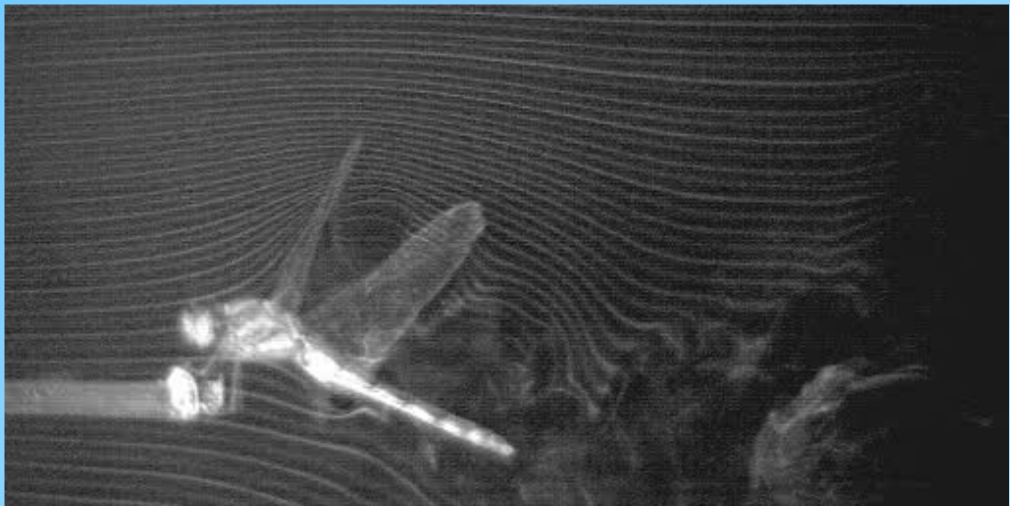
The blue lines show the thin cores of quantized vortices.

Vortices are disordered spatially and temporally.

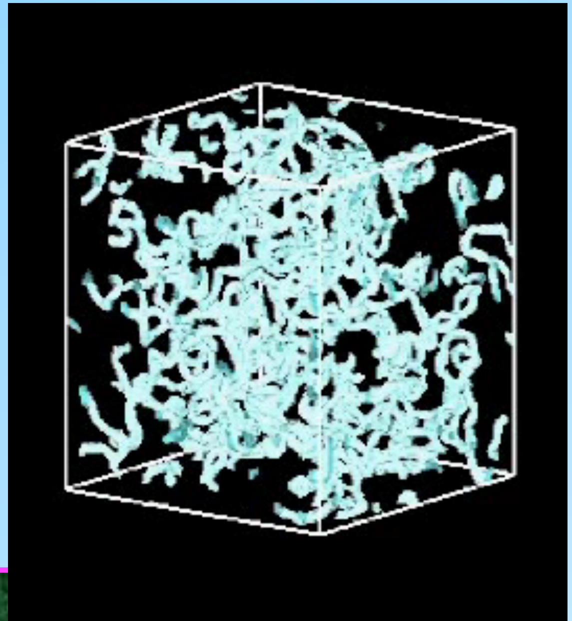
So are superflow created by the vortices.
→ Superfluid turbulence

Classical Turbulence (CT) vs. Quantum Turbulence (QT)

Classical turbulence



Quantum turbulence



Motion of vortex cores

QT may be easier to understand than CT, because each element of turbulence is well defined.

The quantized vortices are
table topological defects.
Every vortex has exactly the
ame circulation.
Circulation is conserved.

What is turbulence?

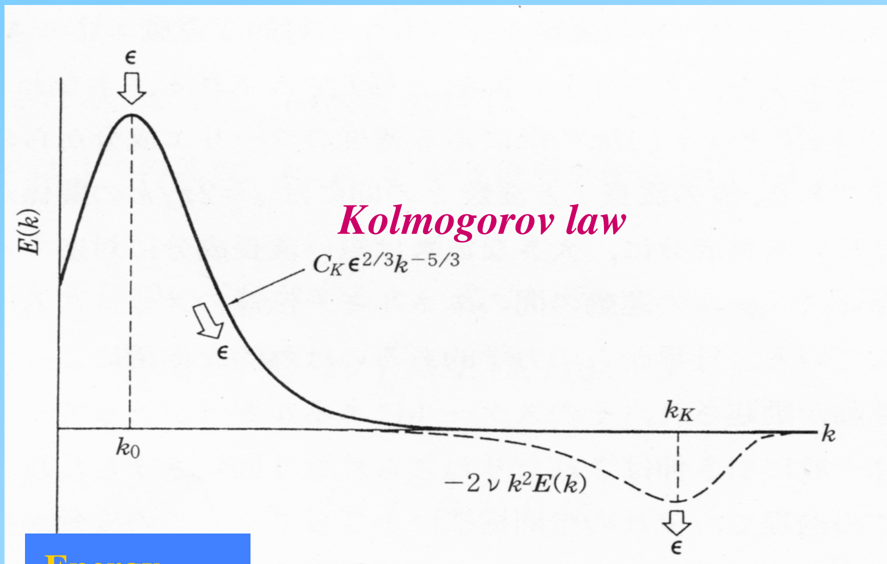
The definition is not unique.
The definition depends on the scientist and the textbook.

<u>Necessary condition</u>
The velocity field is disordered and unpredictable.

<u>Sufficient condition</u>
The disordered physical variables obey some reproducible statistical law.

The Kolmogorov -5/3 spectrum

Disordered velocity field $\boldsymbol{v}(\boldsymbol{r})$



Energy-containing range | **Inertial range** | **Energy-dissipative range**

Energy spectrum of turbulence

Energy spectrum of the velocity field

$$E = \frac{1}{2}\int \mathbf{v}(\mathbf{r})^2 d\mathbf{r} = \int E(k)\,dk$$

Energy-containing range

The energy is injected into the system at $k \sim k_0 \sim 1/L$.

Inertial range

Dissipation does not work. The nonlinear interaction transfers the energy from low *k* region to high *k* region.

Kolmogorov law (K41) : $E(k) = C\varepsilon^{2/3} k^{-5/3}$

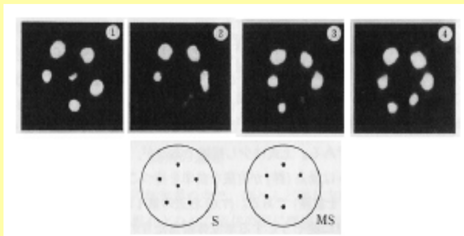
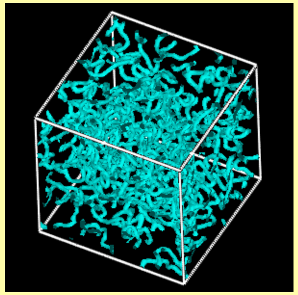
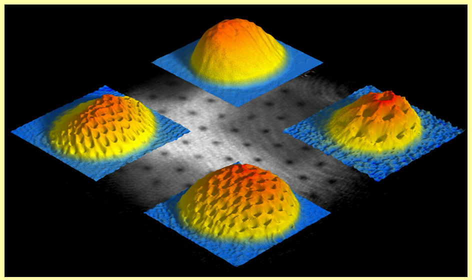
Energy-dissipative range

The energy is dissipated with the rate ε at the Kolmogorov wave number $k_c = (\varepsilon/\nu^3)^{1/4}$.

2. Quantum turbulence in atomic BECs

2-1. QT in single-component BEC

There are two main cooperative phenomena of quantized vortices; Vortex lattice under rotation and Quantum turbulence.

	Vortex lattice	Quantum turbulence
Superfluid He		
Atomic BEC		Few works, but recently active

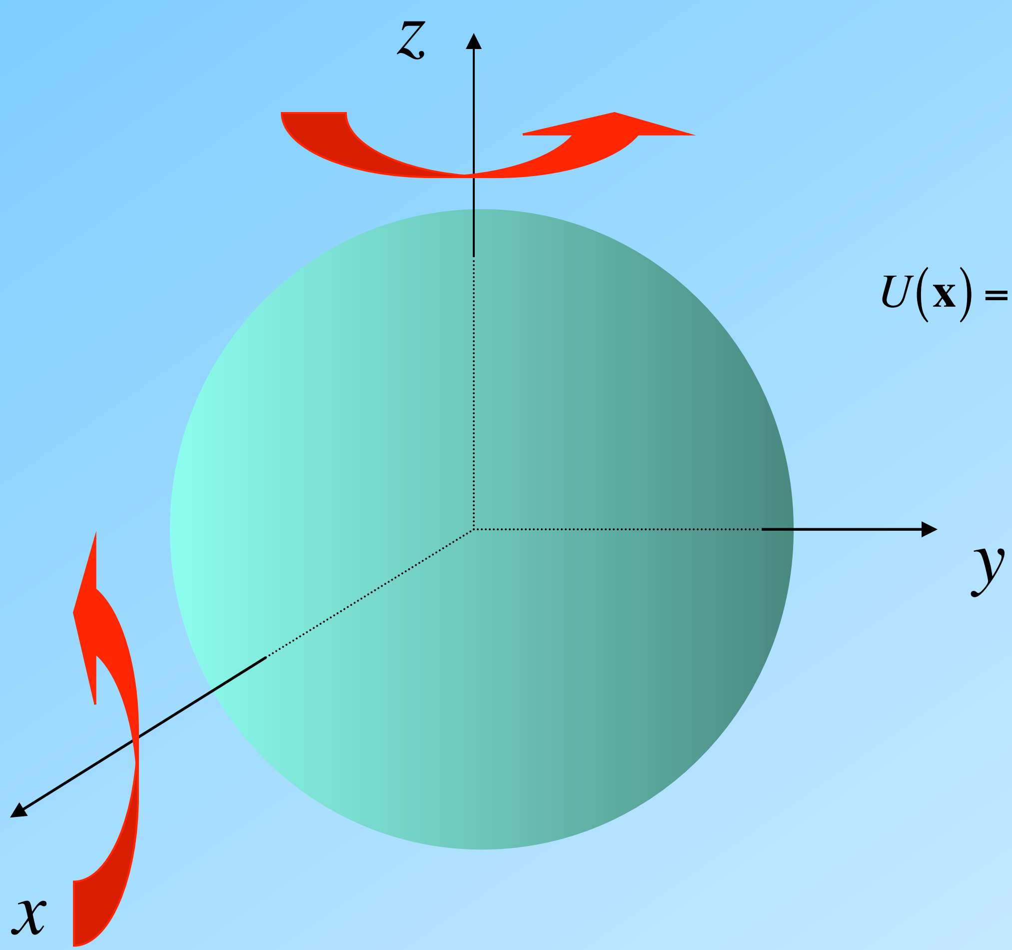
Some methods of how to create QT in a trapped BEC.

- Phase printing: N. G. Berloff, B. V. Svistunov, PRA66, 013603(2002)
- Manipulating the trapping potential: M. Kobayashi, M. Tsubota, PRA76, 045603(2007)
- Stirring the condensate: A. J. Allen *et al*., PRA89, 023602(2014) other several works

QT in a trapped BEC

M. Kobayashi and M. Tsubota, Phys. Rev. A76, 045603 (2007)

Making QT by combining two rotations



1. Trap the BEC in a weakly elliptic potential.

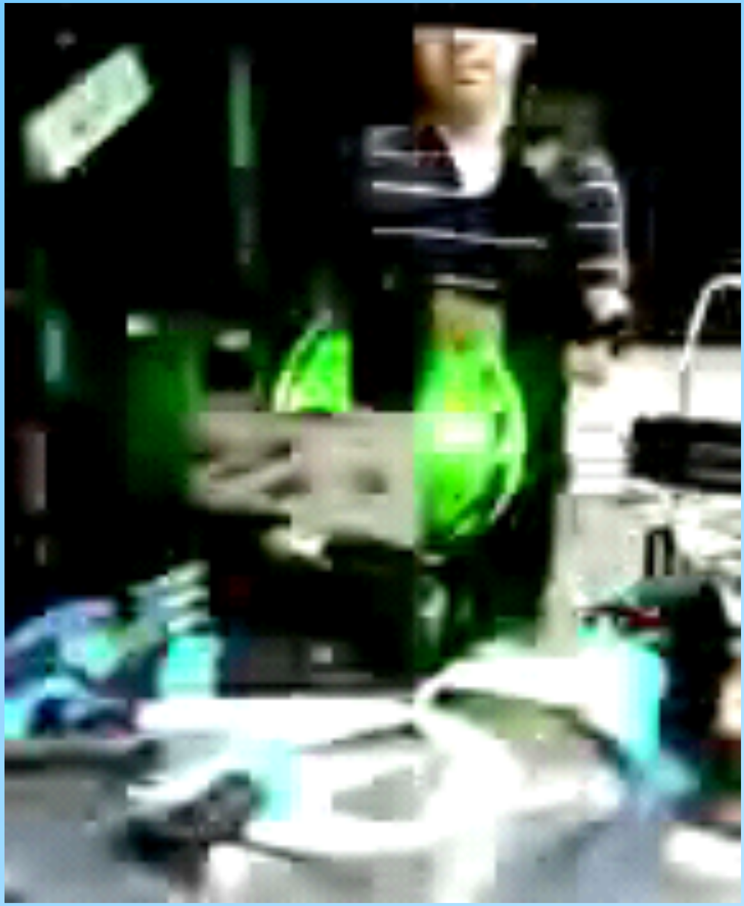
$$U(\mathbf{x}) = \frac{m\omega^2}{2}\left[(1-\varepsilon_1)(1-\varepsilon_2)x^2 + (1+\varepsilon_1)(1-\varepsilon_2)y^2 + (1+\varepsilon_2)z^2\right]$$

2. Rotate the system first around the x-axis, next around the z-axis.

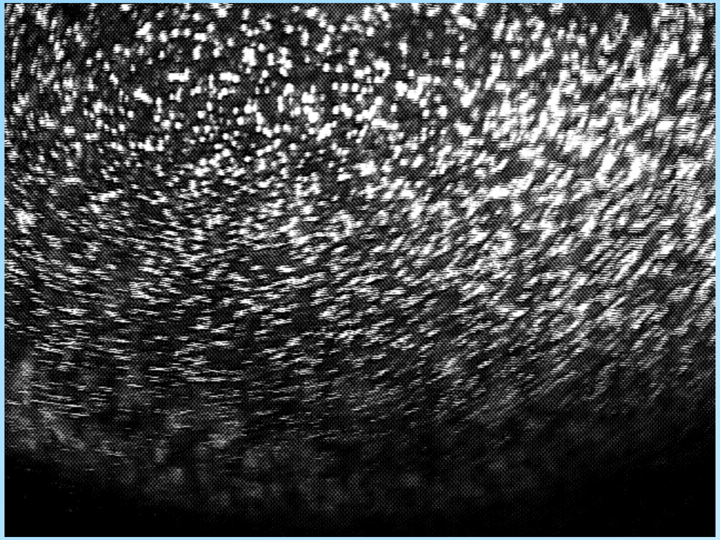
$$\Omega(t) = \left(\Omega_x,\ \Omega_z \sin\Omega_x t, \Omega_z \cos\Omega_x t\right)$$

Actually this idea has been already used in CT.

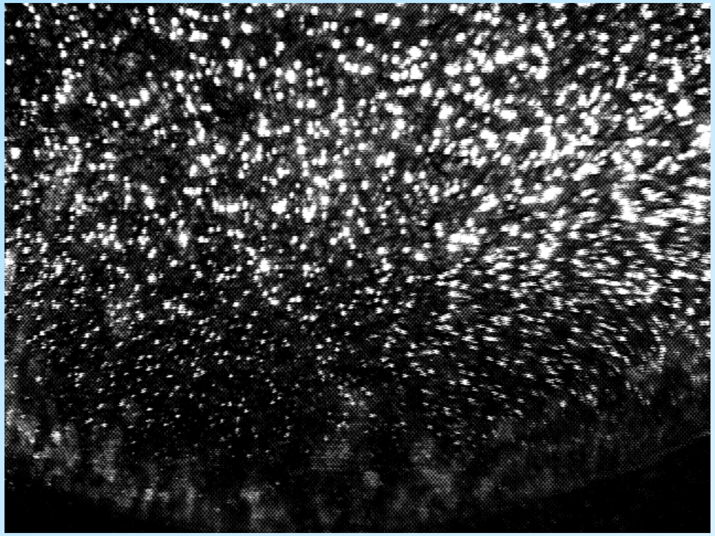
S. Goto, N. Ishii, S. Kida, and M. Nishioka, Phys. Fluids 19, 061705 (2007)



Rotation around one axis



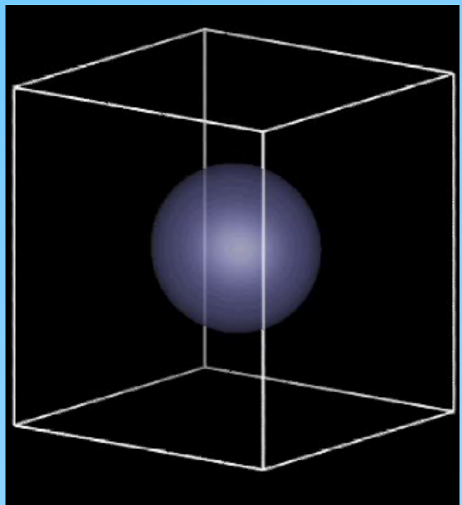
Rotation around two axes



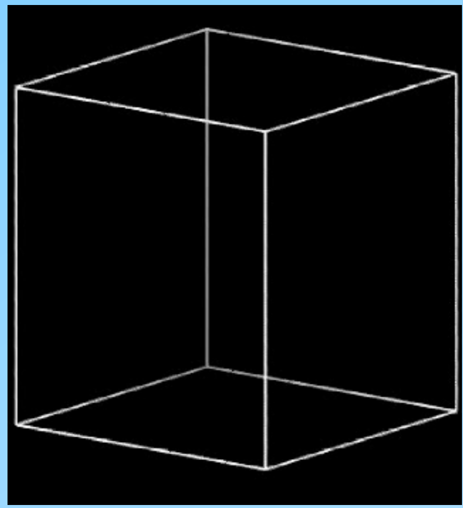
QT made by two precessions in a trapped BEC

Two precessions ($\omega_x \times \omega_z$)

M. Kobayashi and MT, Phys. Rev. A76, 045603 (2007)

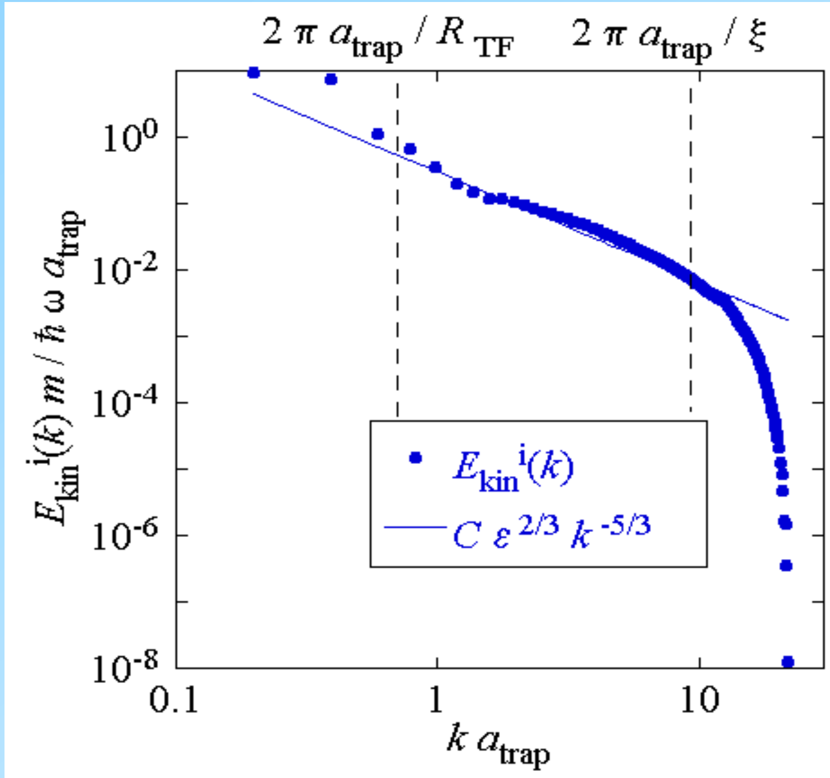


Condensate density



Quantized vortices

Simulation of the Gross-Pitaevskii model

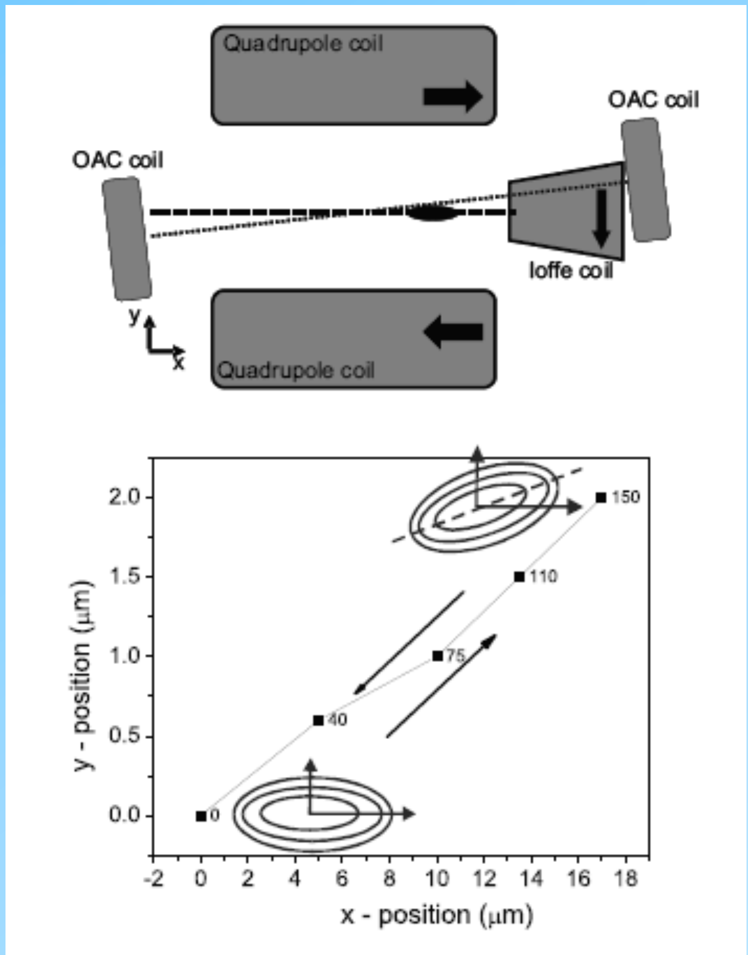


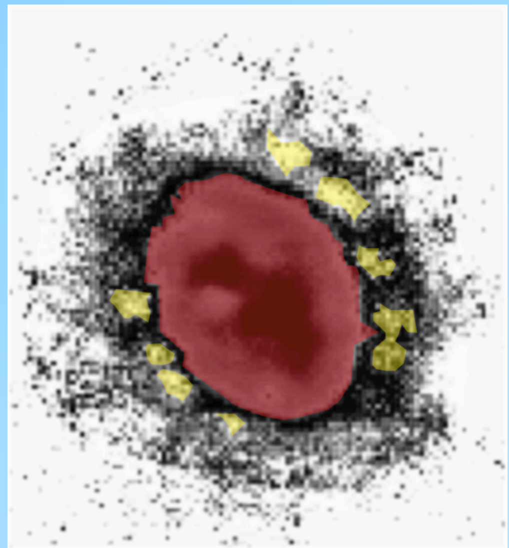
$$n \approx 1.78 \pm 0.194$$

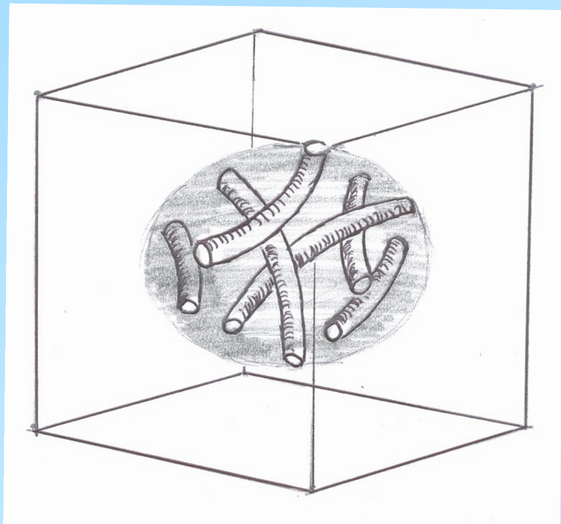
We confirmed a scaling law of the energy spectrum similar to the Kolmogorov -5/3 law.

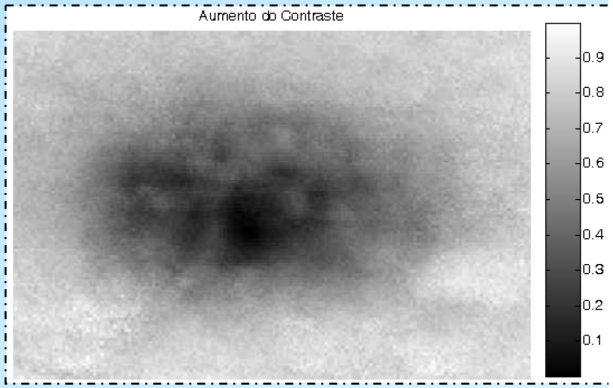
Making 3D QT by exciting a trapped BEC.

E.A.L.Henn *et al.*, PRL103, 045301(2009)







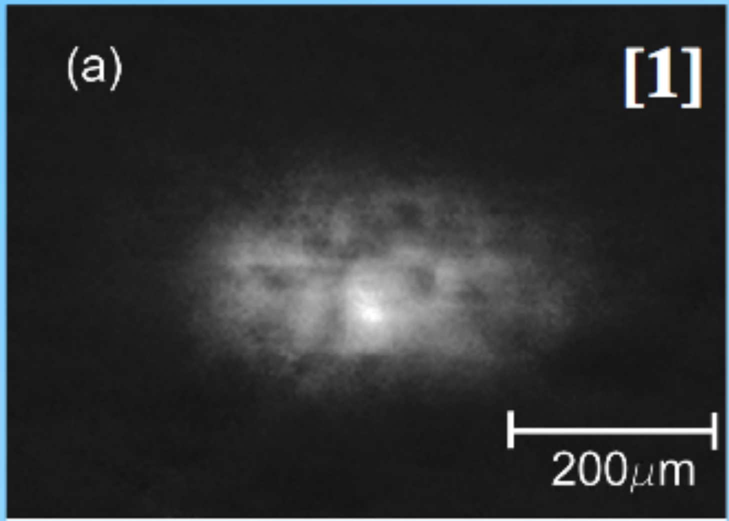


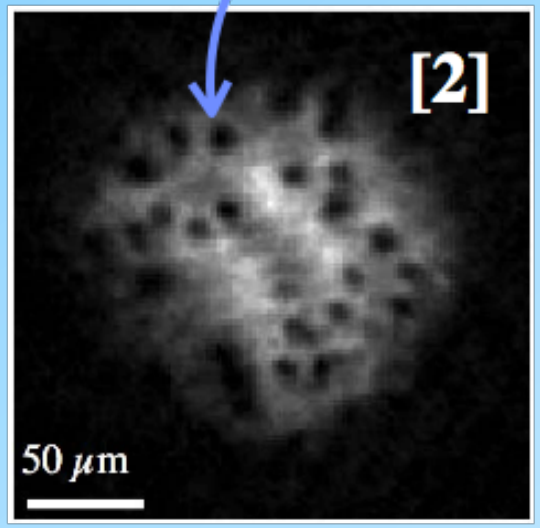
Coupled large amplitude oscillation

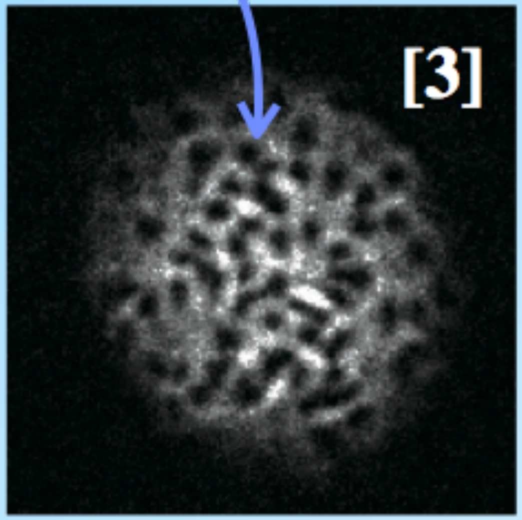
QT is realized experimentally.

- Density distribution in turbulence

Quantized vortex







3D ***2D***

[1] E. A. L. Henn et al. , Phys. Rev. Lett. 103, 045301 (2009).
[2] K. E. Wilson et al., Annu. Rev. Cold At. Mol. 1, 261 (2013).
[3] Woo Jin Kwon et al., Phys. Rev. A 90, 063627 (2014).

2-2. QT in two-component BECs

$$i\hbar\frac{\partial}{\partial t}\Psi_1 = \frac{\mathbf{p}_1^2}{2m_1}\Psi_1 + g_{11}|\Psi_1|^2\Psi_1 + g_{12}|\Psi_2|^2\Psi_1$$

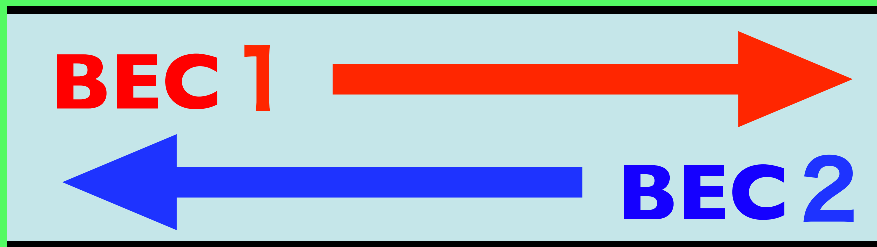
$$i\hbar\frac{\partial}{\partial t}\Psi_2 = \frac{\mathbf{p}_2^2}{2m_2}\Psi_2 + g_{22}|\Psi_2|^2\Psi_2 + g_{12}|\Psi_1|^2\Psi_2$$

g_{11}, g_{22} : intracomponent interaction

g_{12} : intercomponent interaction

$g_{11}g_{22} > g_{12}^2$ → The mixture is stable.

However,



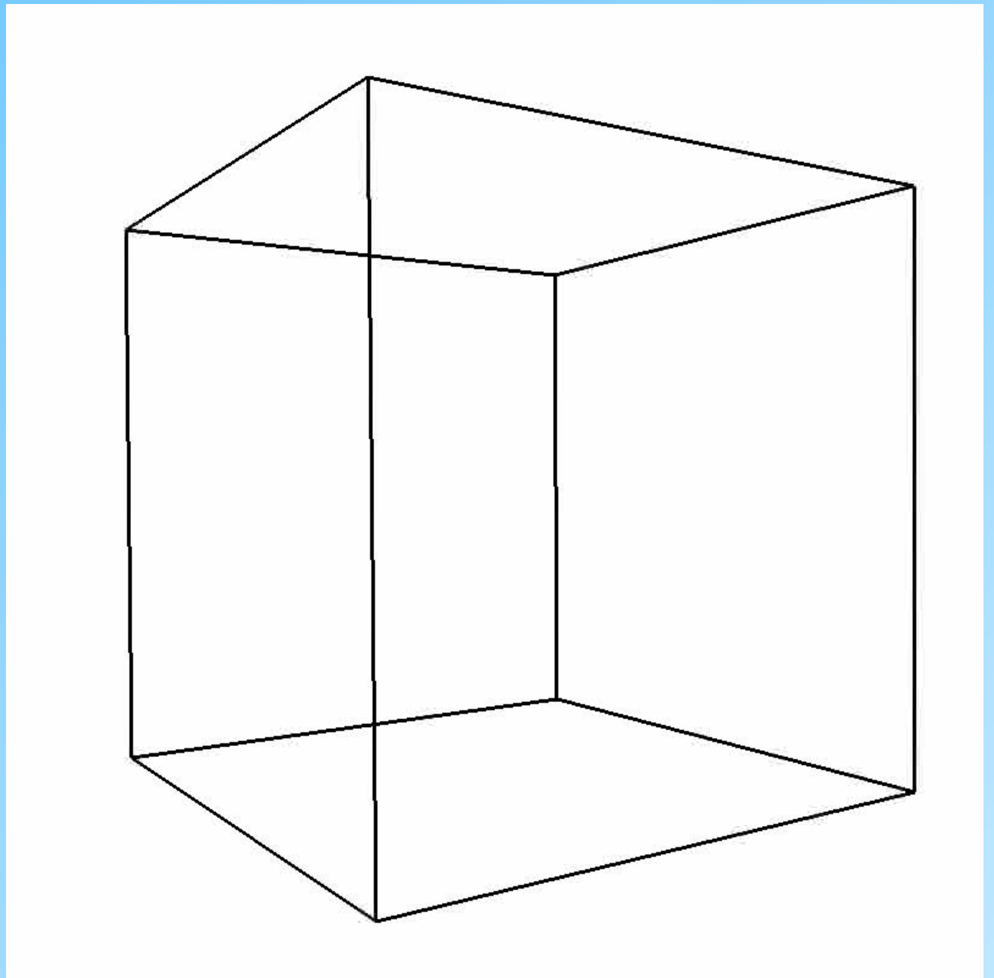
The large relative velocity should make it unstable.

V. I. Yukalov and E. P. Yukalova, Laser Phys. Lett. **1**, 50 (2004).

C. Hamner *et al*., Phys. Rev. Lett. **106**, 065302(2011).

3D 2-component QT

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \quad V'_R = 4.71, \quad \gamma = 0.9$$

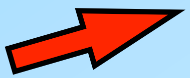


Solitons

→ Vortex loops

→ QT

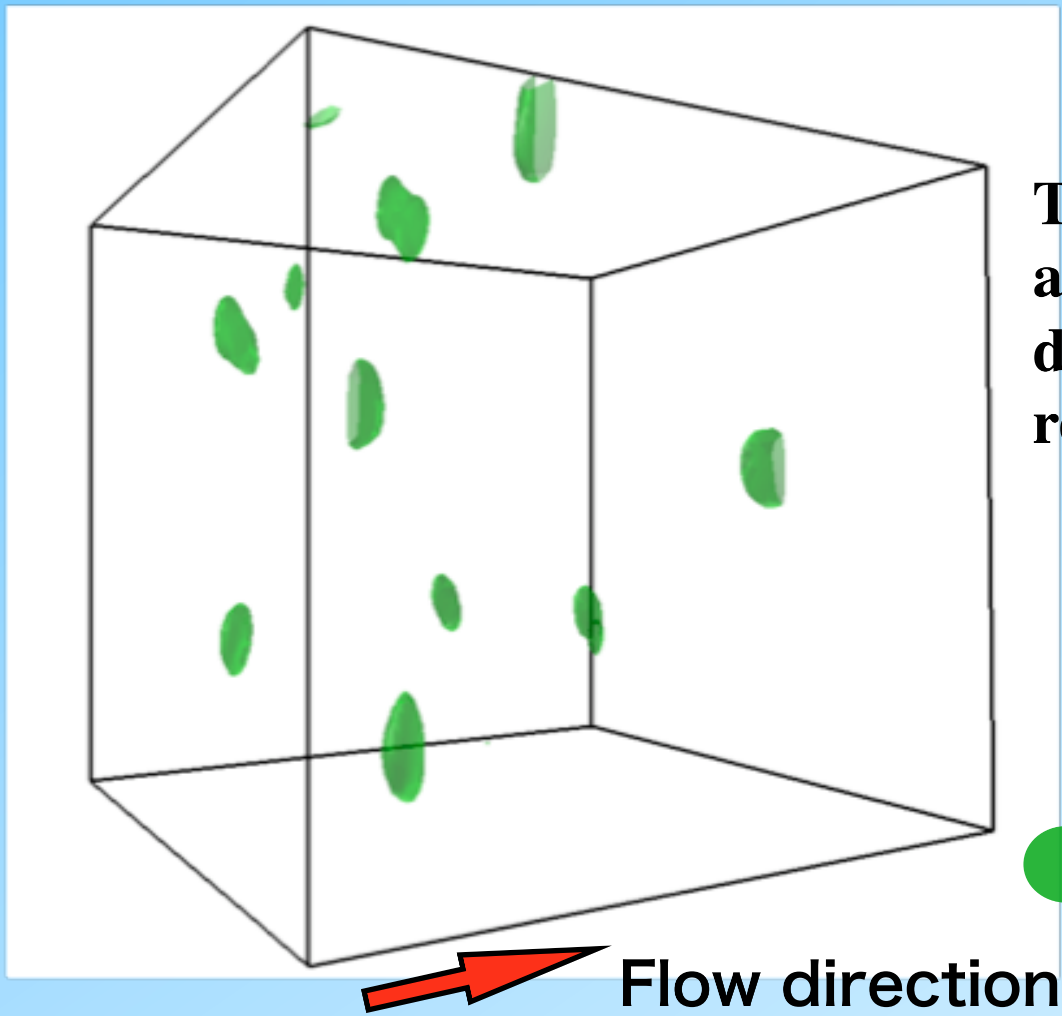
H. Takeuchi, S. Ishino, MT,
PRL105, 205301(2010)
S. Ishino, MT, H. Takeuchi,
PRA83, 063602(2011)



Flow direction

Scenario to turbulence (1)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1,\ \ V_R' = 4.71,\ \ \gamma = 0.9$$

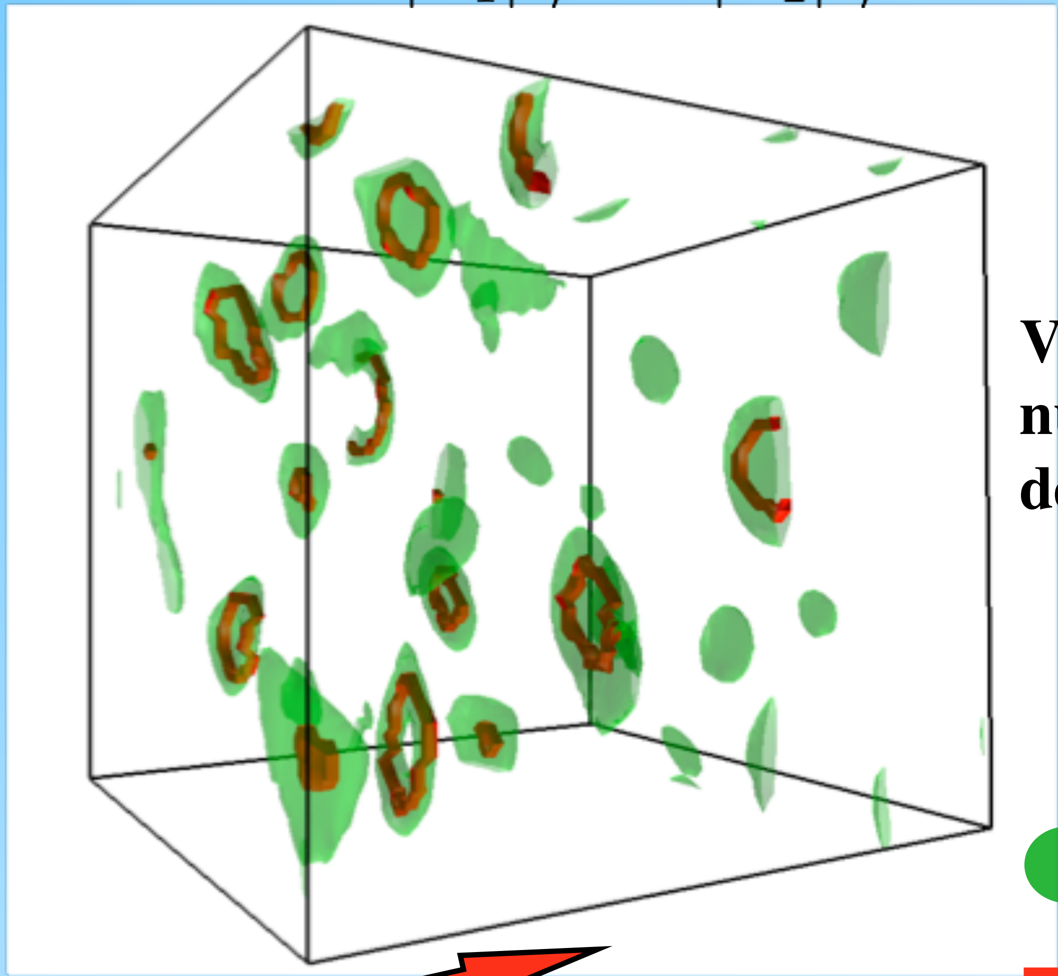


The unstable mode is amplified to lead to the disk-shaped low density regions.

Isosurface of $|\Psi_1|^2/n = 0.1$

Scenario to turbulence (2)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \;\; V_R' = 4.71, \;\; \gamma = 0.9$$



Vortex rings are nucleated inside the low density regions.

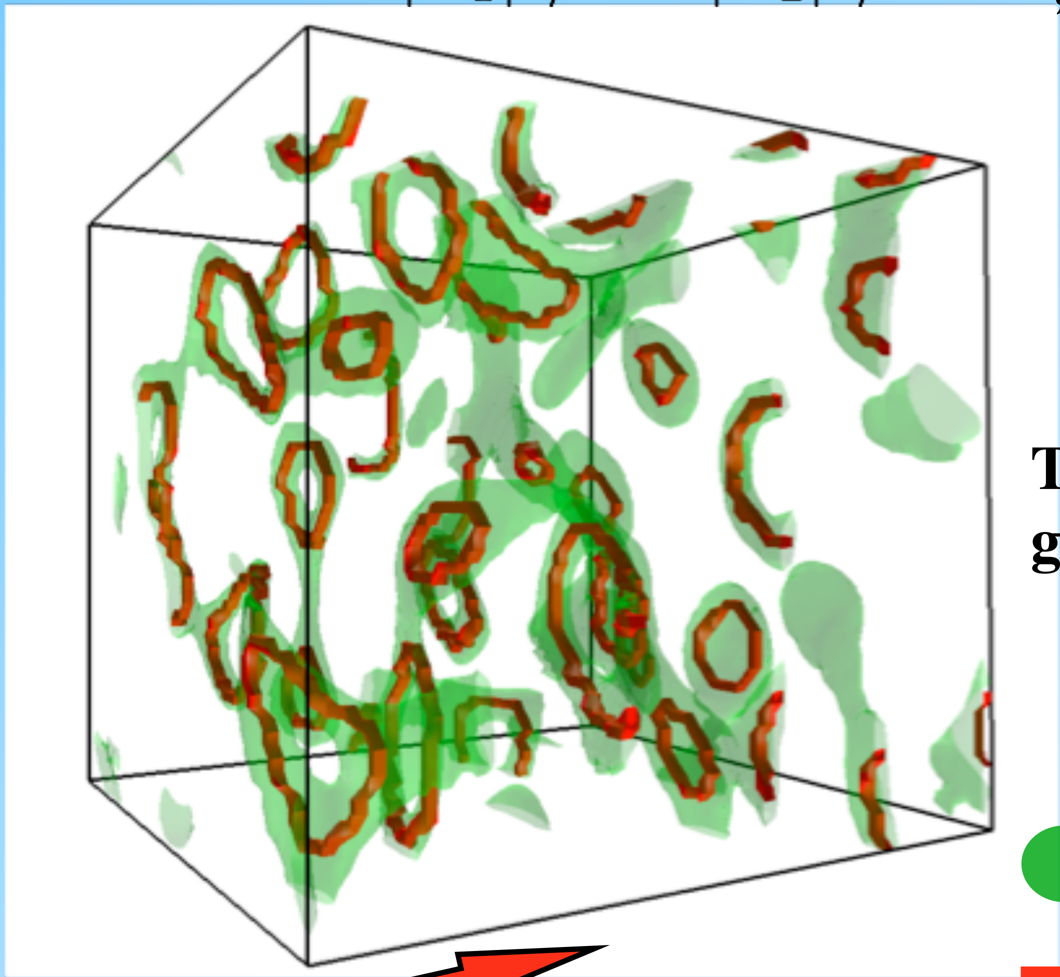
Isosurface of $|\Psi_1|^2/n = 0.1$

Vortex core of component 1

Flow direction

Scenario to turbulence (3)

$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \ V_R' = 4.71, \ \gamma = 0.9$



The vortices expand and grow.

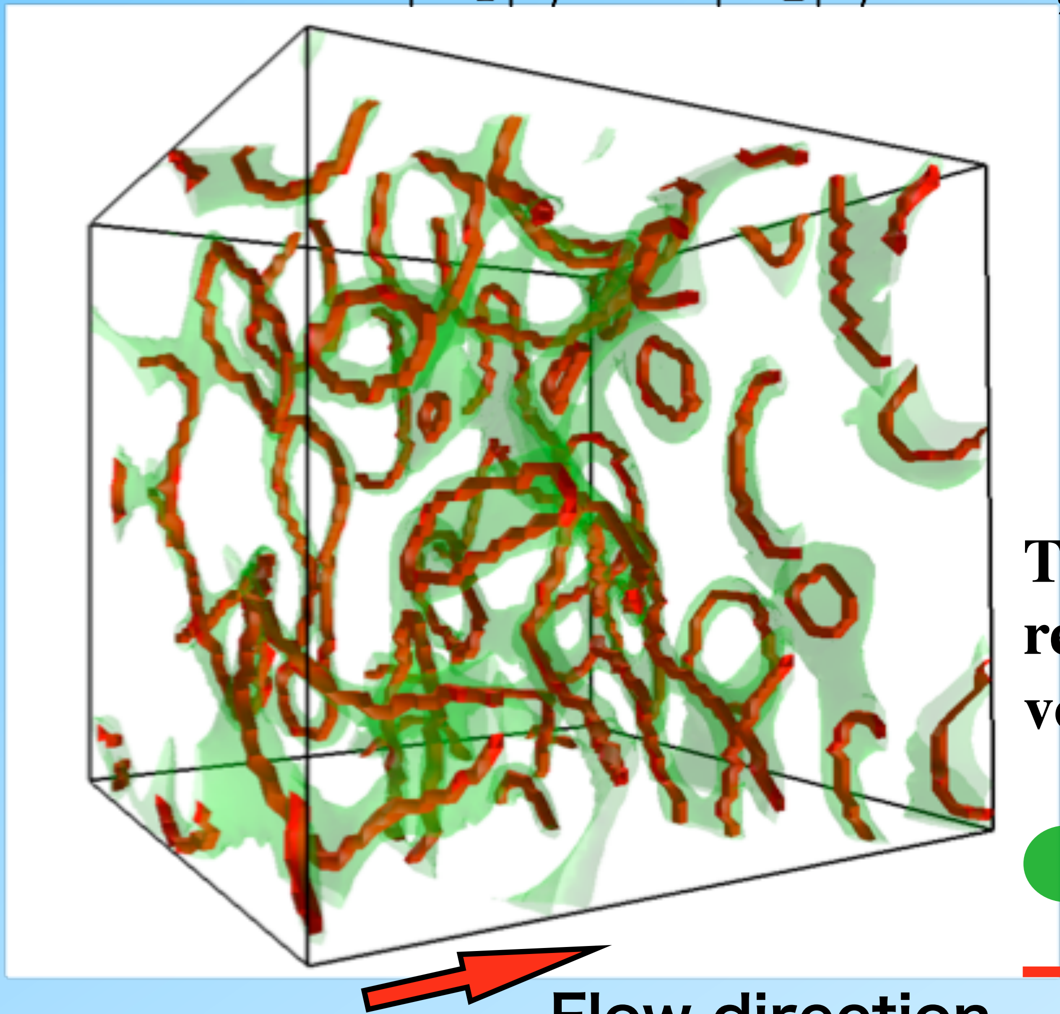
Flow direction

Isosurface of $|\Psi_1|^2/n = 0.1$

Vortex core of component 1

Scenario to turbulence (4)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \quad V_R' = 4.71, \quad \gamma = 0.9$$



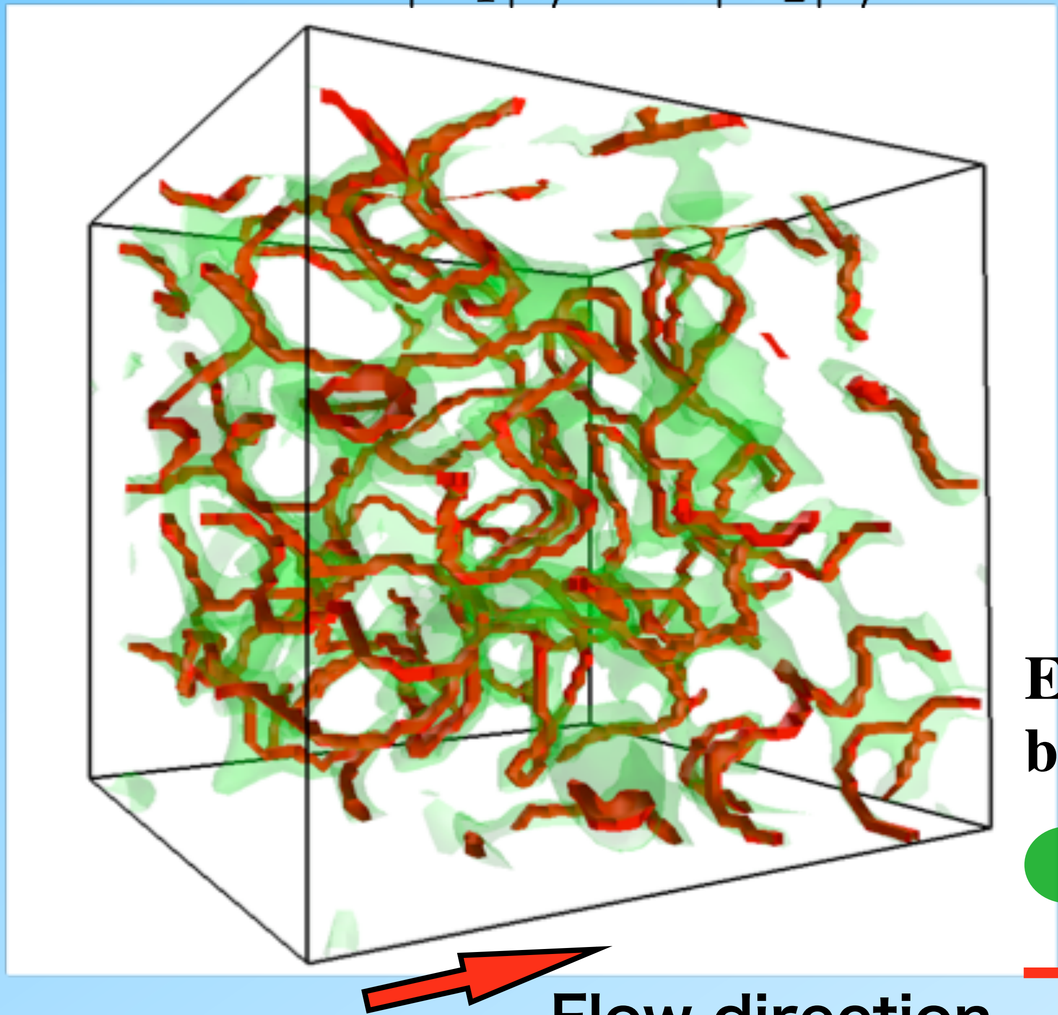
The vortices expand to reconnect with other vortices.

Isosurface of $|\Psi_1|^2/n = 0.1$

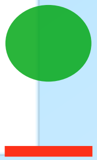
Vortex core of component 1

Scenario to turbulence (5)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \quad V_R' = 4.71, \quad \gamma = 0.9$$

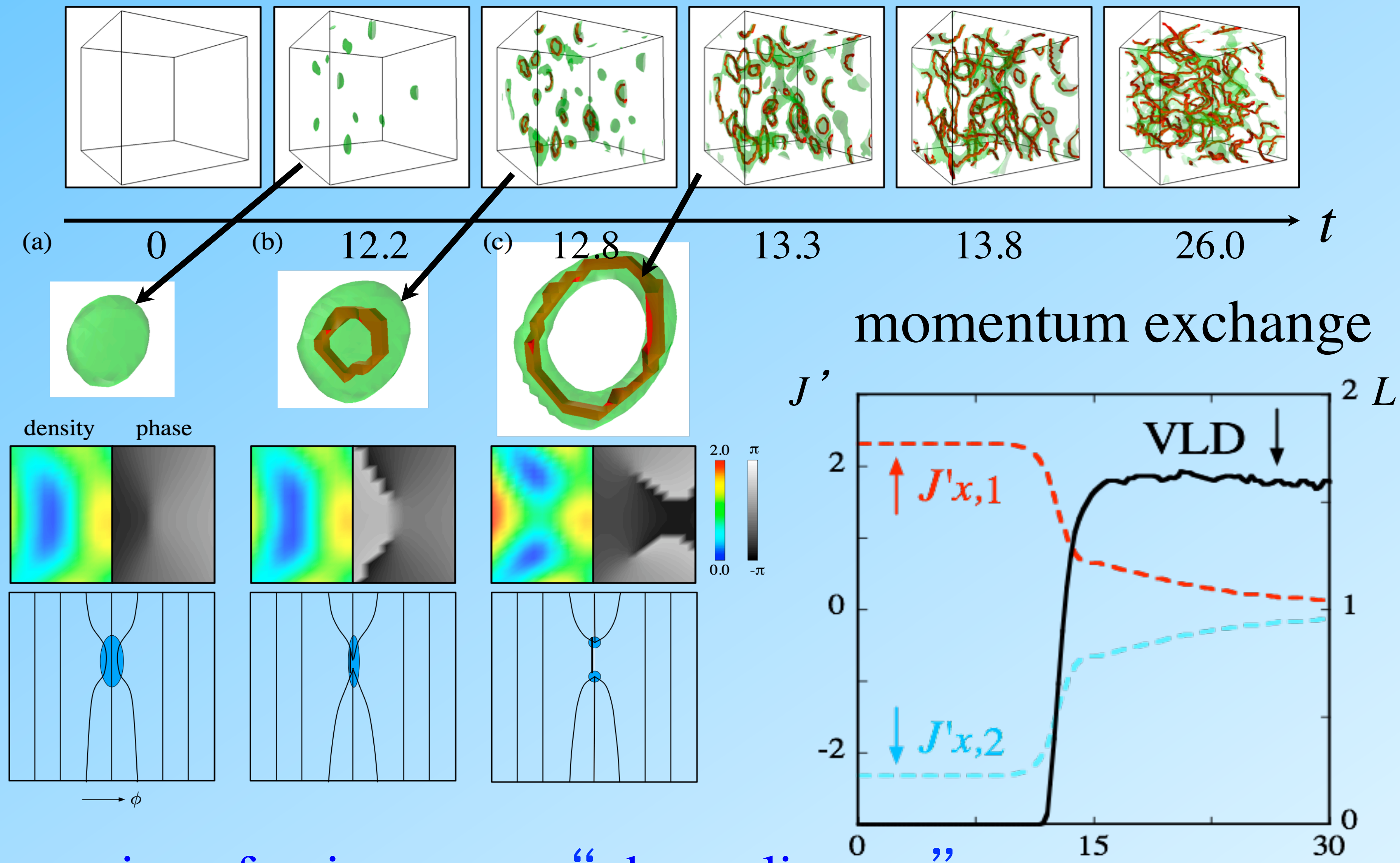


Eventually the vortices become tangled.



Flow direction

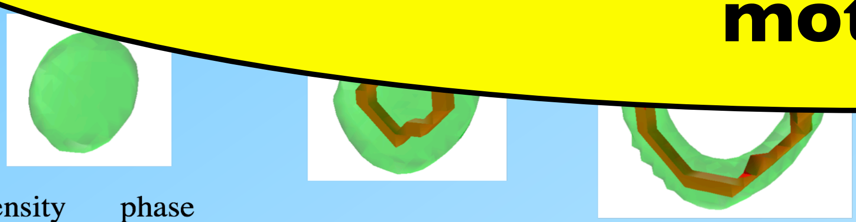
Scenario to turbulence

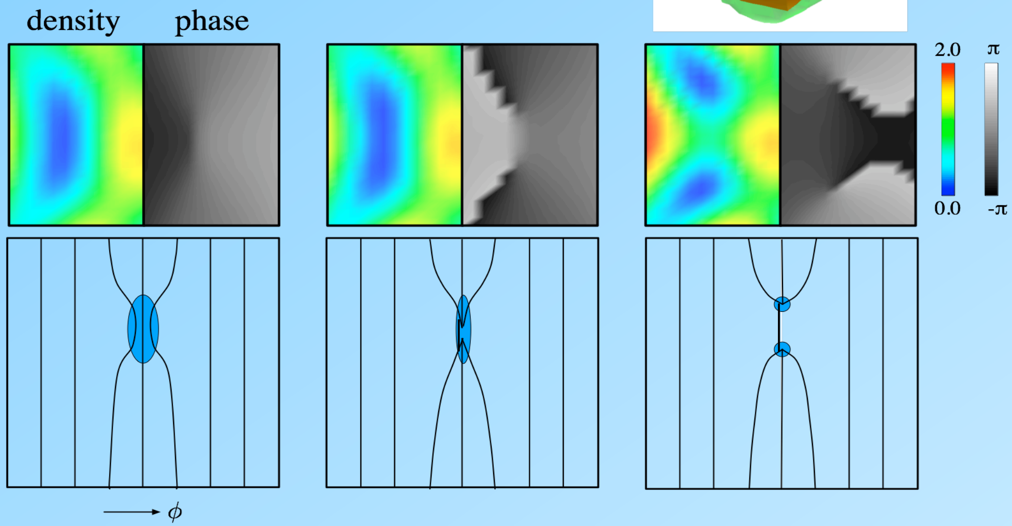


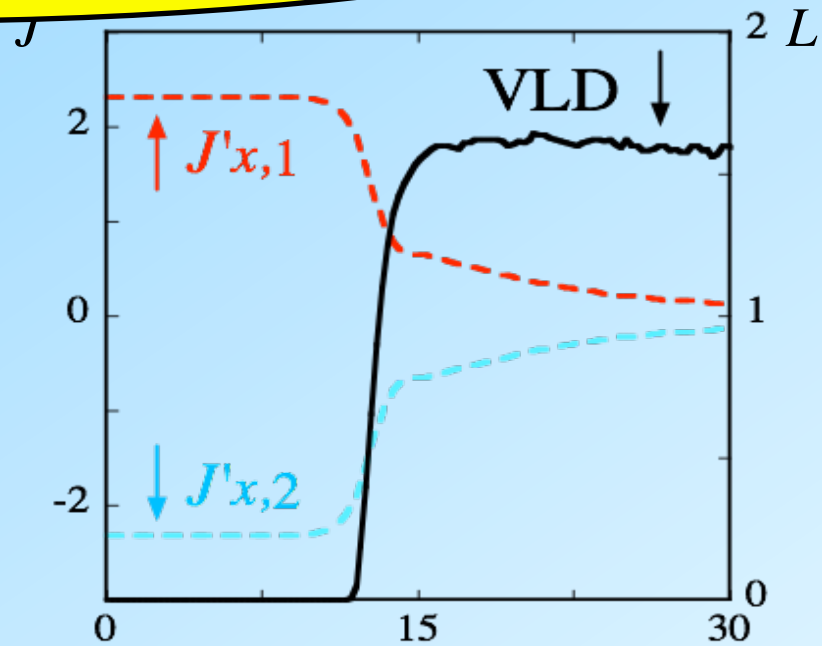
Expansion of a ring means “phase slippage”.

Scenario to turbulence

The "mutual friction" between two condensates exchange momentum between them to reduce their relative motion.

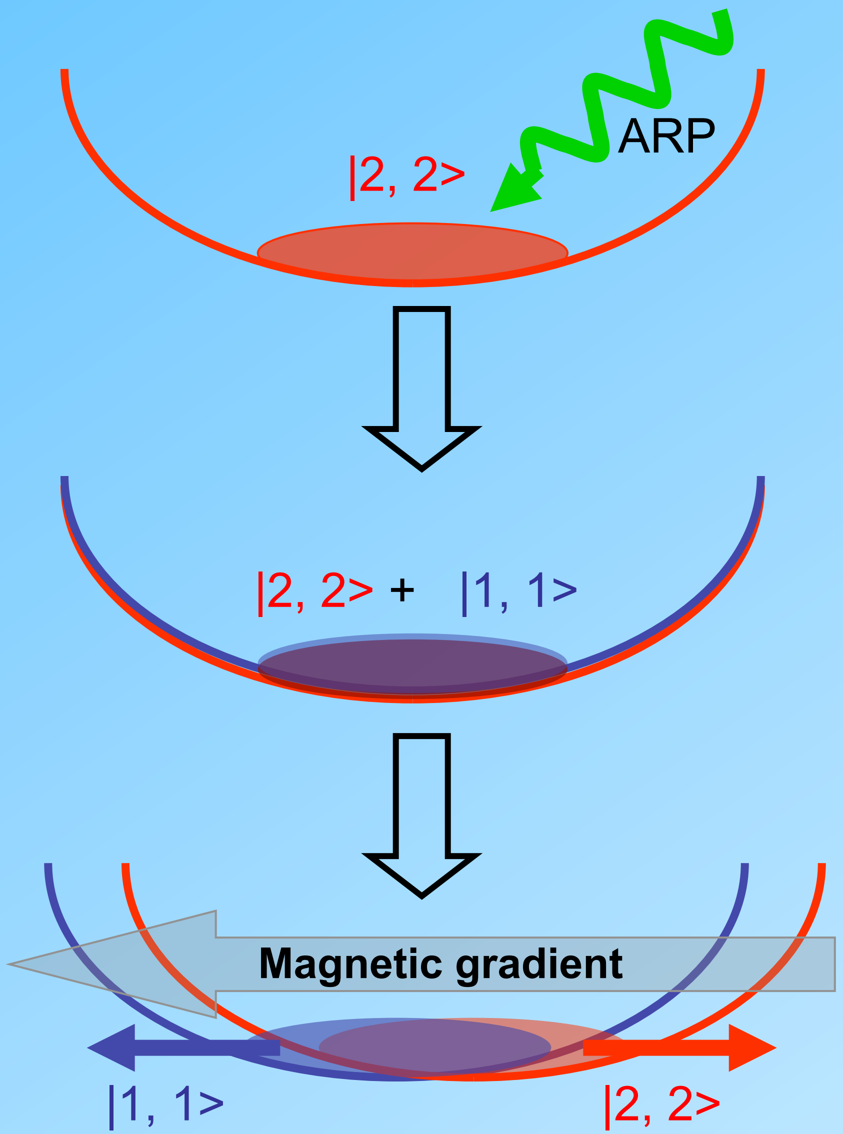


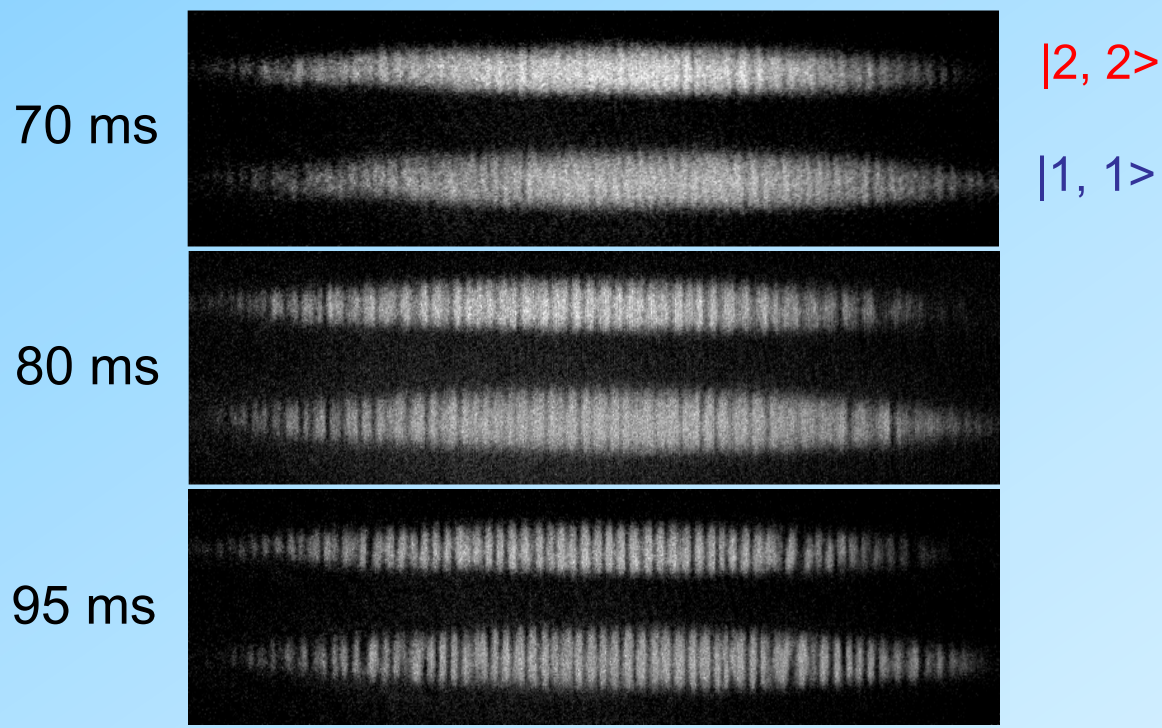




Counterflow is realized in ^{87}Rb.

C. Hamner *et al*., Phys. Rev. Lett. **106**, 065302(2011).

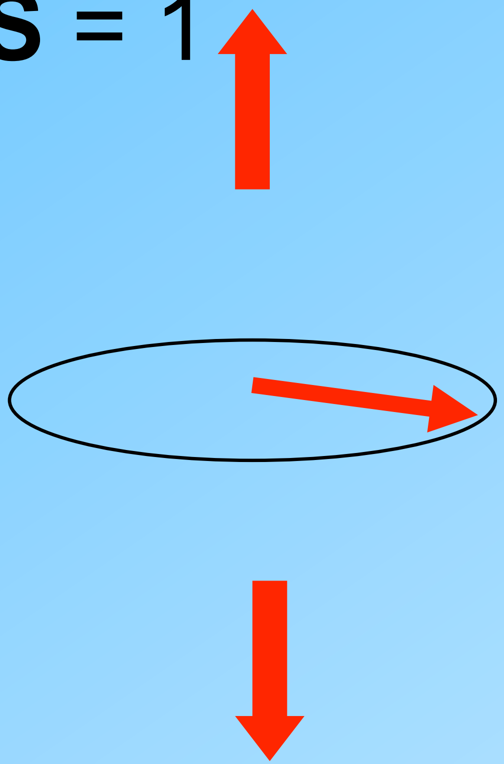




The modulation instability due to counterflow is observed!

2-3. Spin turbulence in spinor BECs

S = 1



$m = 1 \quad \psi_1$

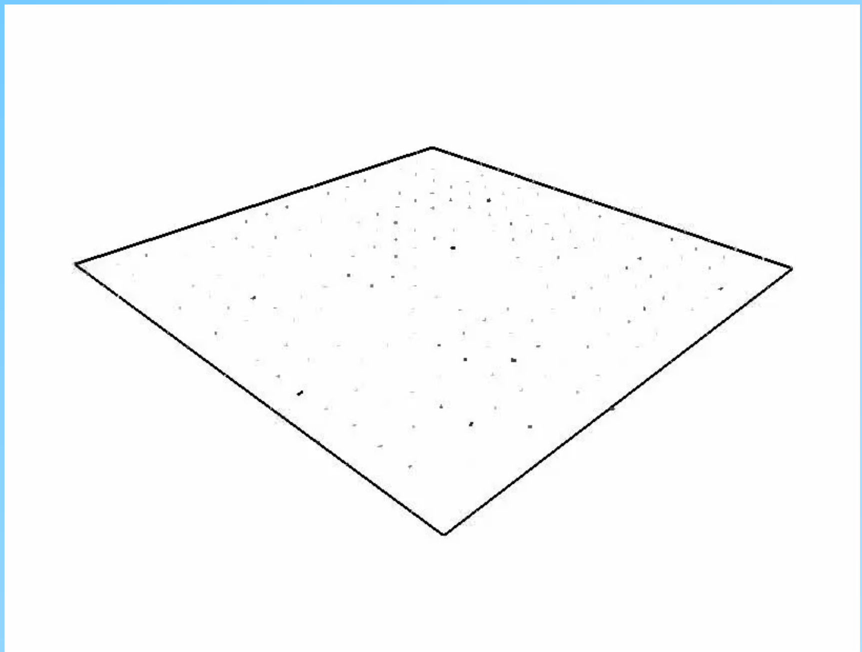
$m = 0 \quad \psi_0$

$m = -1 \quad \psi_{-1}$

Spin-1 spinor Bose-Einstein condensate

^{23}Na, ^{87}Rb, etc.

Spin turbulence(ST) in spinor BECs



Motion of the spin density vectors

Spins are disordered spatially and temporally.

How to characterize ST?

What is the order parameter of ST?

1. Energy spectrum of ST

K. Fujimoto, MT: PRA85, 033642(2012),
PRA85, 053641(2012)

2. Analogy with spin glass

MT, Y. Aoki, K. Fujimoto: PRA88, 061601(R)
(2013)

Spin-1 spinor Gross-Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\psi_m = -\frac{\hbar^2}{2M}\nabla^2\psi_m + c_0 n\psi_m + c_1\sum_{m=-1}^{1}\mathbf{s}\cdot\mathbf{S}_{mn}\psi_n \qquad (m = 1,0,-1)$$

Condensate density: $n = \sum_{m=-1}^{1}|\psi_m|^2$

Spin density vector: $s_i = \sum_{m,n=-1}^{1}\psi_m^*(S_i)_{mn}\psi_n$

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\quad S_y = \frac{i}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\quad S_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

Total energy

$$E = \int\sum_{m=-1}^{1}\left[\psi_m^*\left(-\frac{\hbar^2}{2M}\nabla^2\right)\psi_m\right]d\mathbf{r} + \frac{c_0}{2}\int n^2 d\mathbf{r} + \underline{\frac{c_1}{2}\int \mathbf{s}^2 d\mathbf{r}}$$

Spin-dependent interaction energy

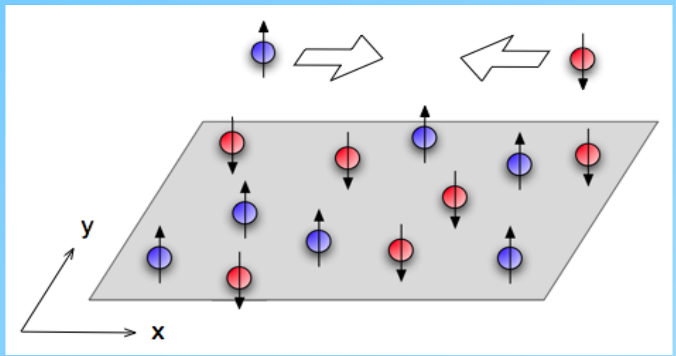
Here we confine ourselves to the case of the ferromagnetic interaction $c_1 < 0$.

1. Energy spectrum of ST

K. Fujimoto, MT: PRA85, 033642(2012), PRA85, 053641(2012)

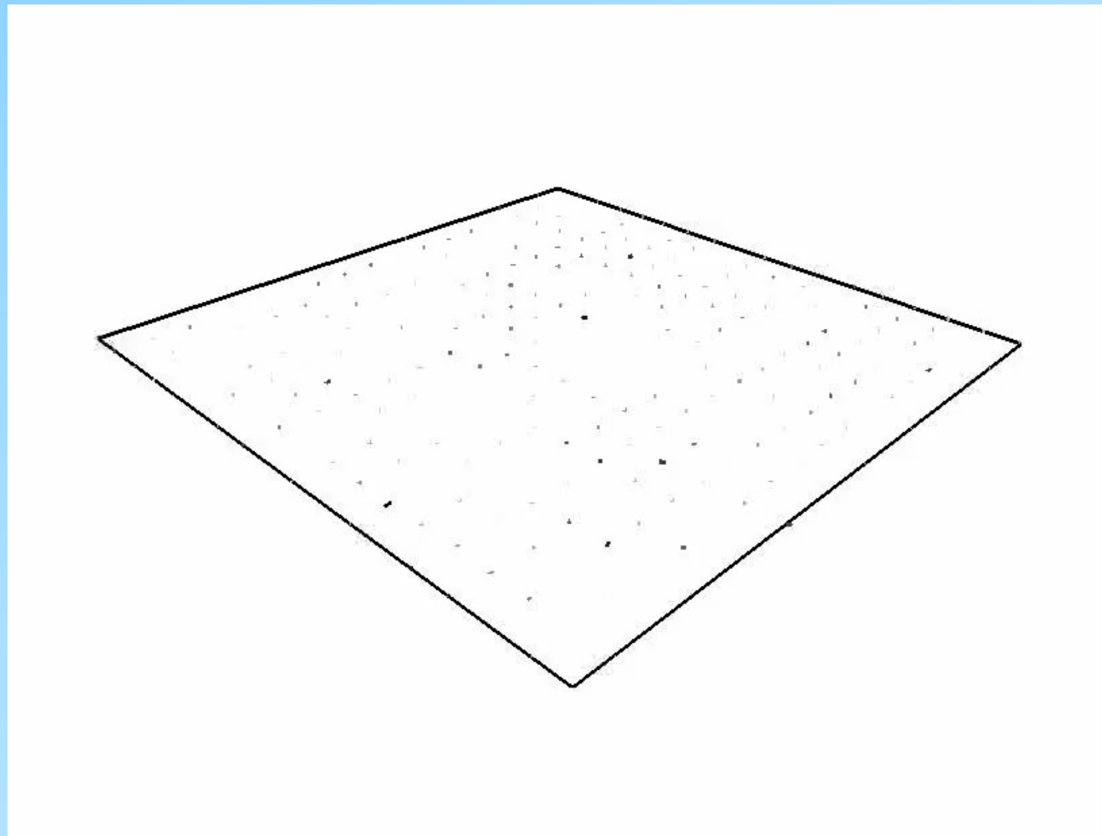
$$s_i = \sum_{m,n=-1}^{1} \psi_m^* (S_i)_{mn} \psi_n$$

m=1 *m*= -1



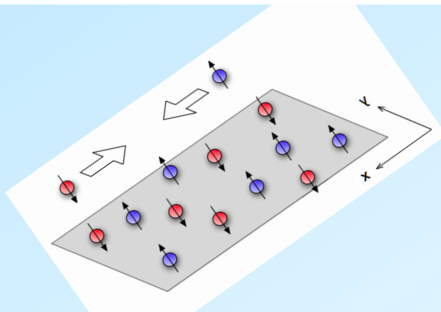
Counterflow between *m*=±1 components

Time-development of the spin density vector **s**

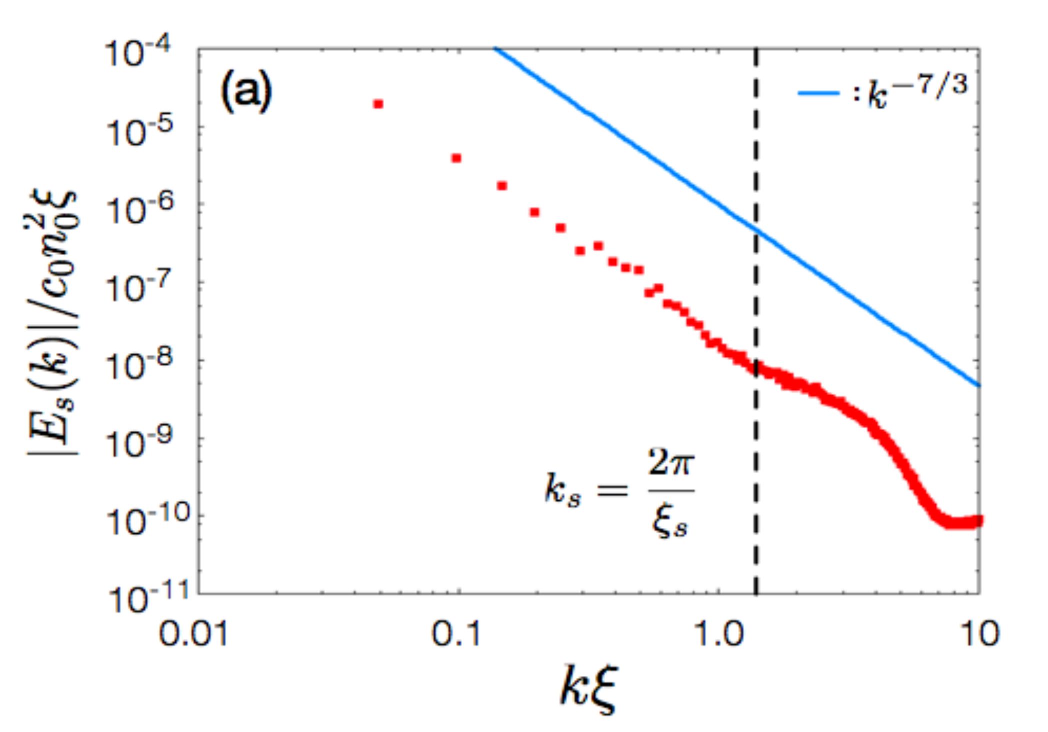


The counterflow makes the system unstable towards spin turbulence.

Spin turbulence (ST)



Spectrum of the spin-dependent interaction energy in the turbulent state



$$E_s = \frac{c_1}{2A}\int \mathbf{s}(\mathbf{r})^2 d\mathbf{r}, \quad \mathbf{s}(\mathbf{r}) = \sum_{\mathbf{k}} \tilde{\mathbf{s}}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})$$

$$E_s = \frac{c_1}{2}\sum_{\mathbf{k}} \left|\tilde{\mathbf{s}}(\mathbf{k})\right|^2, \quad E_s(k) = \frac{c_1}{2\,\Delta k}\sum_{k<|\mathbf{k}_1|<k+\Delta k} \left|\tilde{\mathbf{s}}(\mathbf{k}_1)\right|^2$$

$$E_s \propto k^{-7/3}$$

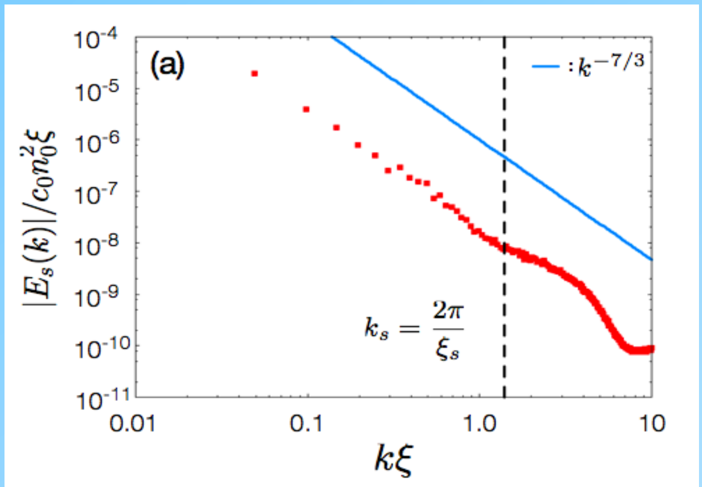
ξ_s: spin coherence length

This power law is understood through the dimensional scaling analysis of the equation of the motion of **s**.

Dimensional Scaling analysis of obtaining the -7/3 law (1)

The Kolmogorov -5/3 law is obtained from the analysis of the Navie-Stokes(NS) equation.

NS equation: $$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{v}$$ T. Watanabe *et al*., PRE55, 5575(1997).



GP equations → Equation of spin

$$\frac{\partial}{\partial t}\hat{\mathbf{s}} + (\mathbf{v}\cdot\nabla)\hat{\mathbf{s}} = \frac{\hbar}{2m}\hat{\mathbf{s}}\times\left[\nabla^2\hat{\mathbf{s}} + \left(\frac{\nabla n}{n}\cdot\nabla\right)\hat{\mathbf{s}}\right], \quad \hat{\mathbf{s}} \equiv \frac{\mathbf{s}}{n}$$

K. Kudo and Y. Kawaguchi, PRA84, 043607(2011)

In a uniform system, we can assume ∇n is negligible and $\boldsymbol{v}$ is small.

The Fourie component of $\hat{\mathbf{s}}$ obeys

$$\frac{\partial}{\partial t}\hat{\mathbf{s}}(\mathbf{k},t) = \sum_{\mathbf{k}_1,\mathbf{k}_2}\mathbf{k}_2^2\,\hat{\mathbf{s}}(\mathbf{k}_1,t)\times\hat{\mathbf{s}}(\mathbf{k}_2,t)\delta_{\mathbf{k},\mathbf{k}_1+\mathbf{k}_2}$$

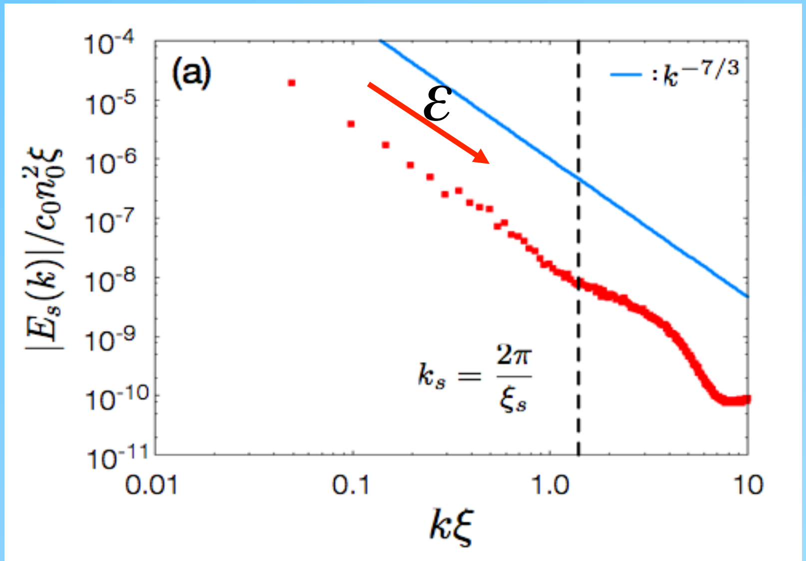
We request this equation is invariant under the scale transformation

$$\mathbf{k}\to\alpha\mathbf{k}, \quad t\to\beta t$$

Then $\hat{\mathbf{s}}$ must be transformed like $\hat{\mathbf{s}} \to \alpha^{-2}\beta^{-1}\hat{\mathbf{s}}$, $\boxed{\hat{\mathbf{s}} \approx k^{-2}t^{-1}}$

Dimensional Scaling analysis of obtaining the -7/3 law (2)

$$\hat{\mathbf{s}} \approx k^{-2} t^{-1}$$



We request that the energy flux

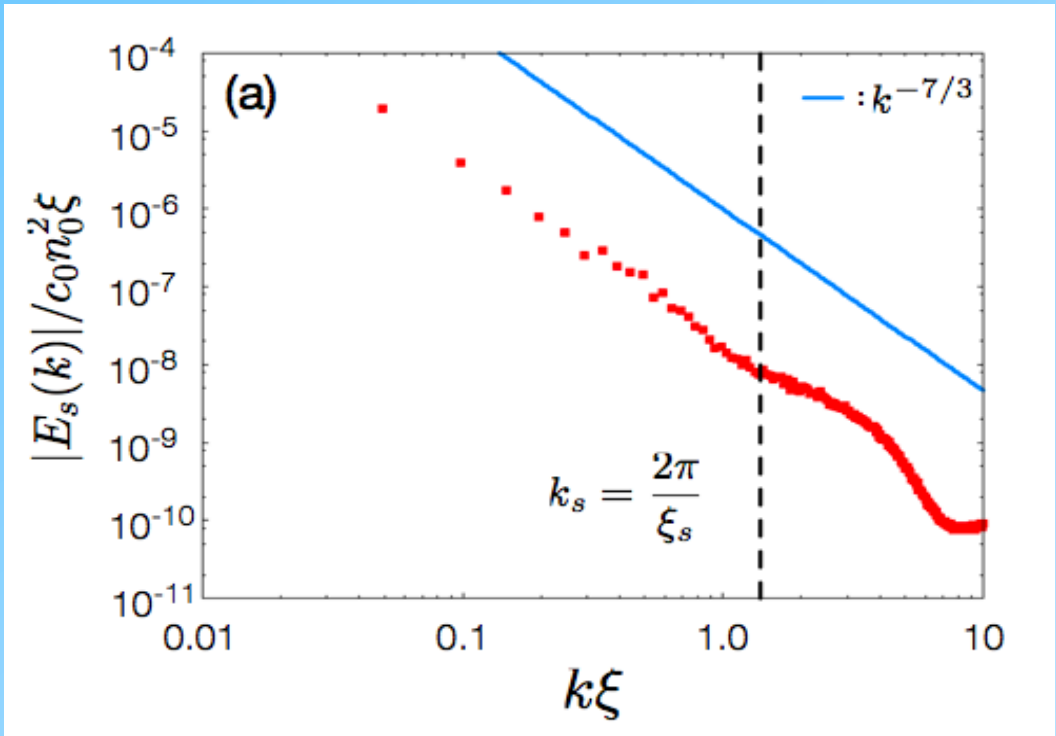
$$\varepsilon \approx \frac{E_s}{t} \approx \frac{\hat{\mathbf{s}}^2}{t} \approx k^{-4} t^{-3} \approx \text{constant}$$

$$\rightarrow \quad t \approx \varepsilon^{-1/3} k^{-4/3}$$

$$\Rightarrow \quad E_s \approx \frac{\hat{\mathbf{s}}^2}{k} \approx k^{-5} t^{-2} \approx \varepsilon^{2/3} k^{-7/3}$$

We obtained a novel -7/3 power law spectrum in ST.

Spectrum of the spin-dependent interaction energy in the turbulent state



$$E_s = \frac{c_1}{2A}\int \mathbf{s}(\mathbf{r})^2 d\mathbf{r}, \quad \mathbf{s}(\mathbf{r}) = \sum_{\mathbf{k}} \tilde{\mathbf{s}}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})$$

$$E_s = \frac{c_1}{2}\sum_{\mathbf{k}} \left|\tilde{\mathbf{s}}(\mathbf{k})\right|^2, \quad E_s(k) = \frac{c_1}{2\,\Delta k}\sum_{k<|\mathbf{k}_1|<k+\Delta k} \left|\tilde{\mathbf{s}}(\mathbf{k}_1)\right|^2$$

$$E_s \propto k^{-7/3}$$

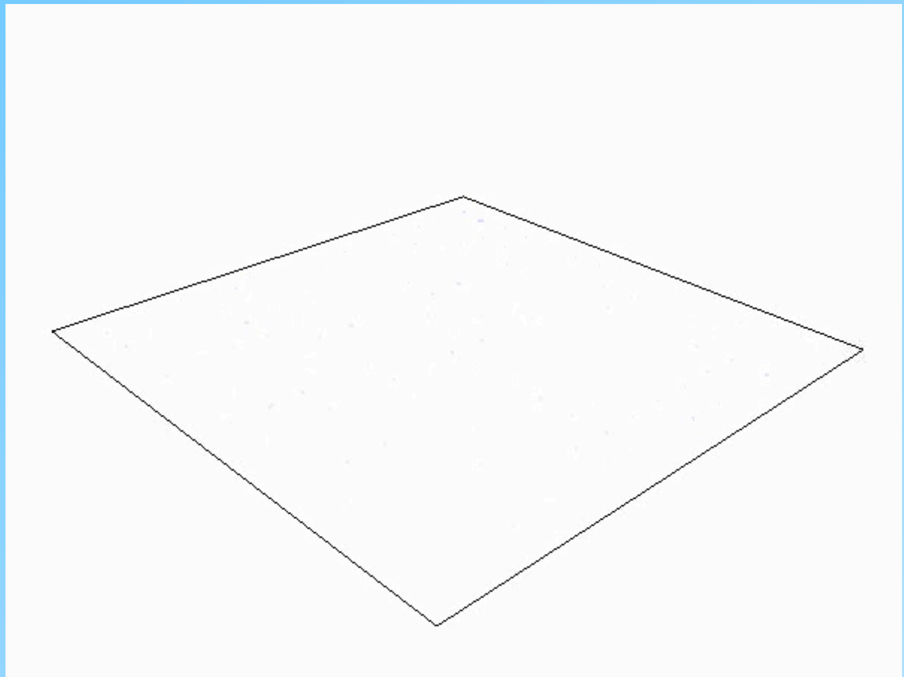
ξ_s: spin coherence length

This power law is understood through the dimensional scaling analysis of the equation of the motion of **s**.

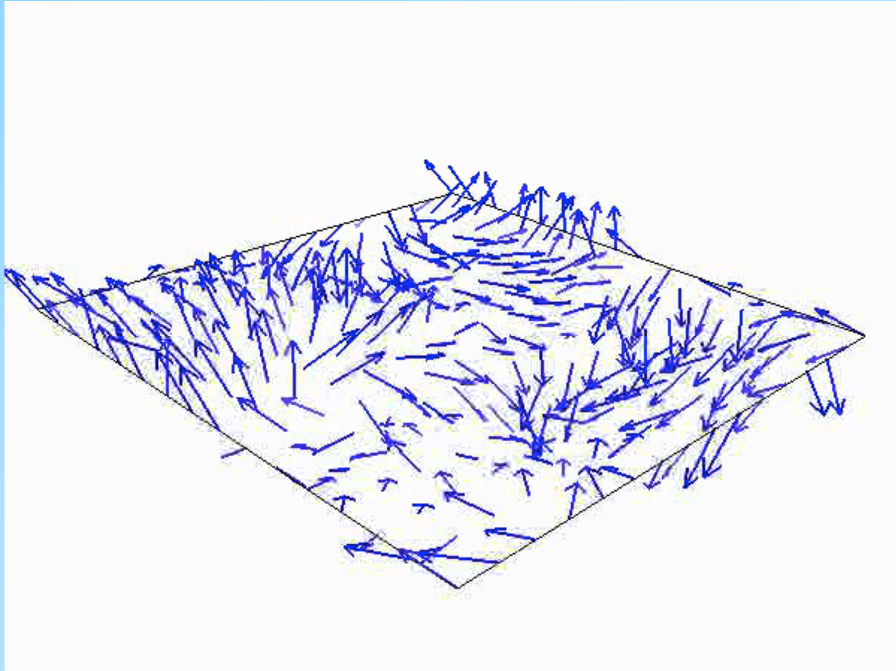
2. Analogy with spin glass

MT, Y. Aoki, K. Fujimoto: PRA88, 061601(R)(2013)

Counterflow instability in a uniform system



Early stage of ST

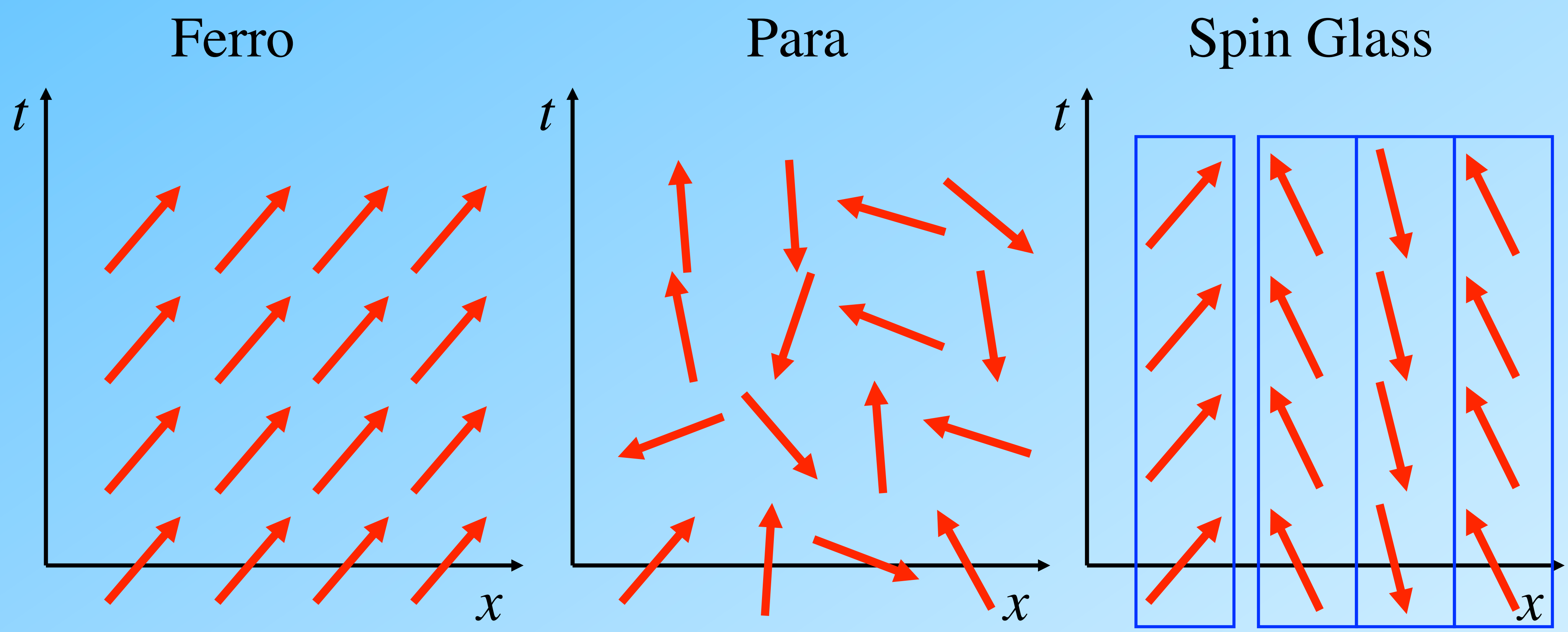


Late stage of ST after the -7/3 law appeared

In the late stage, spins are spatially disordered but temporally frozen.

---> ***Analogy of Spin glass!?***

How to characterize the frozen spins? (1)



Such an order parameter was introduced in the field of spin glass.

D. Sherrington and S. Karkpatrick, PRL35, 1972 (1975)

How to characterize the frozen spins? (2)

D. Sherrington and S. Karkpatrick, PRL35, 1972 (1975)

For the spin variable $\mathbf{S}_i(t)$ in a lattice system

Time average $< \boldsymbol{S}_i(t) >= \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{S}_i(t)$

Spatial average $[\boldsymbol{S}_i(t)] = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{S}_i(t)$

Defining two order parameters

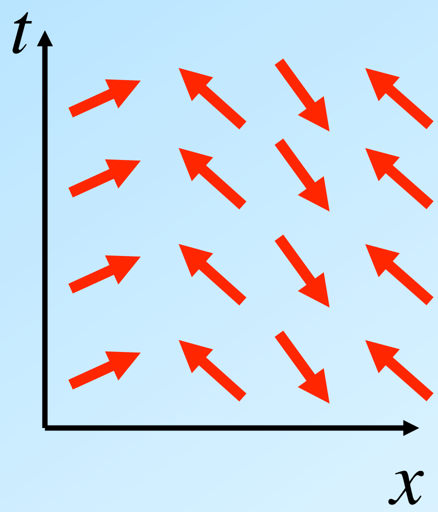
Magnetization

$$\mathbf{m} = \left[\left\langle \mathbf{S}_i(t)\right\rangle\right] = \frac{1}{N}\sum_{i=1}^{N}\left\langle \mathbf{S}_i(t)\right\rangle$$

Spin glass order parameter

$$q = \left[\left\langle \mathbf{S}_i(t)\right\rangle^2\right] = \frac{1}{N}\sum_{i=1}^{N}\left\langle \mathbf{S}_i(t)\right\rangle^2$$

	Para	Ferro	Spin glass
m	0	$\neq 0$	0
q	0	$\neq 0$	$\neq 0$



How to characterize the frozen spins? (3)

Defining the unit vector $\hat{\mathbf{s}}(\mathbf{r},t)=\mathbf{s}(\mathbf{r},t)/|\mathbf{s}(\mathbf{r},t)|$ in order to focus on the spin direction.

Spatial average: $\left[\hat{\mathbf{s}}(\mathbf{r},t)\right]=\dfrac{1}{A}\int_A \hat{\mathbf{s}}(\mathbf{r},t)\,d\mathbf{r}$ A: area of the system

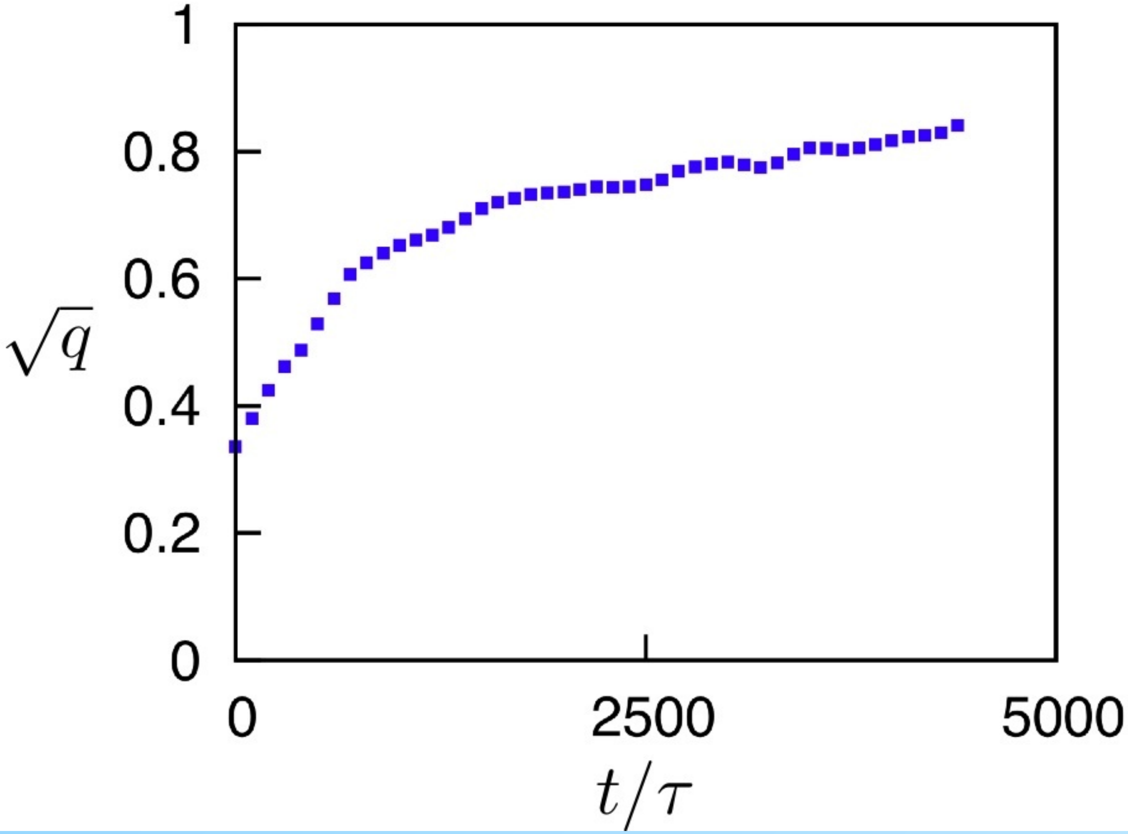
Time average during [t, $t+T$]: $\left\langle \hat{\mathbf{s}}(\mathbf{r},t)\right\rangle_T=\dfrac{1}{T}\int_t^{t+T}\hat{\mathbf{s}}(\mathbf{r},t_1)\,dt_1$

Our time-dependent order parameter:

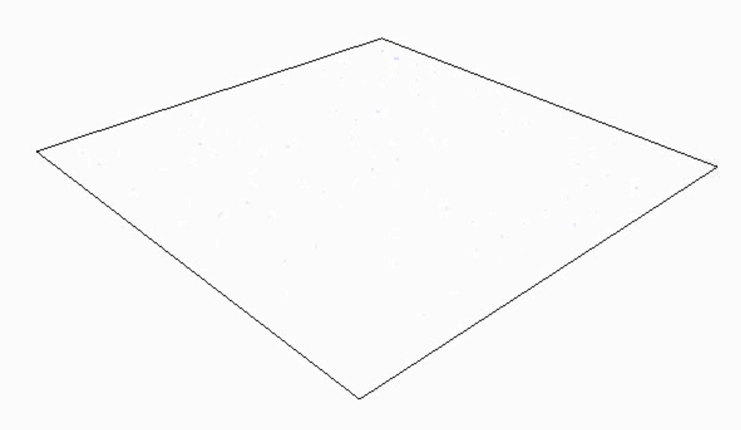
$$q(t)=\left[\left\langle \hat{\mathbf{s}}(\mathbf{r},t)\right\rangle_T^2\right]$$

Spin turbulence in spinor BECs

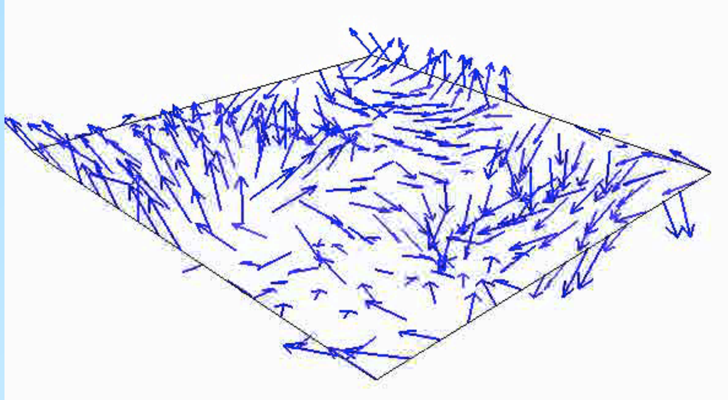
MT, Y. Aoki, K. Fujimoto, PRA88, 061601(R)(2013)



The growth of *q* means that the spins are frozen.



t /τ = 60～360

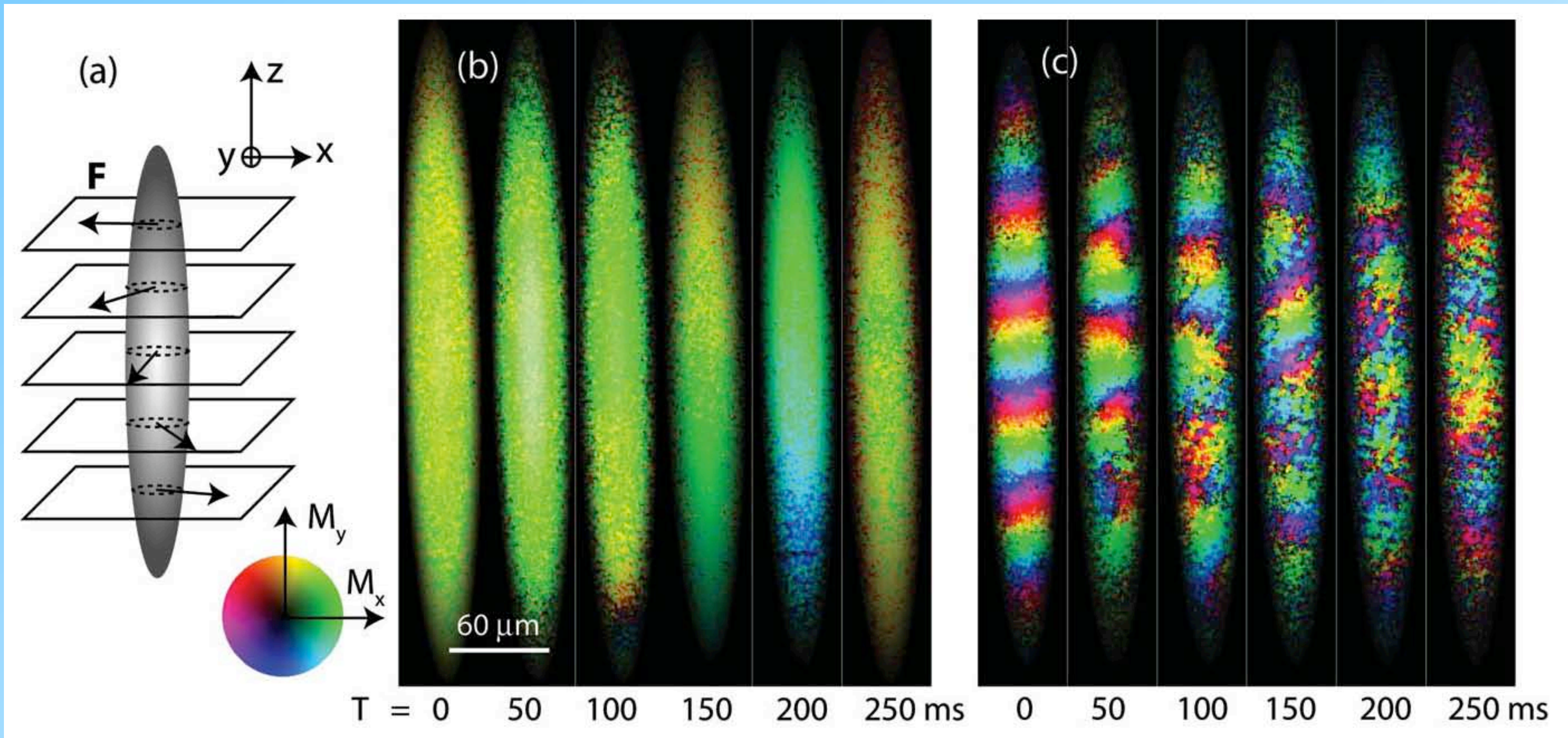


t /τ = 4000～4600

PRL 100, 170403 (2008) PHYSICAL REVIEW LETTERS week ending 2 MAY 2008

Spontaneously Modulated Spin Textures in a Dipolar Spinor Bose-Einstein Condensate

M. Vengalattore,[1] S. R. Leslie,[1] J. Guzman,[1] and D. M. Stamper-Kurn[1,2]
[1]*Department of Physics, University of California, Berkeley, California 94720, USA*
[2]*Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*



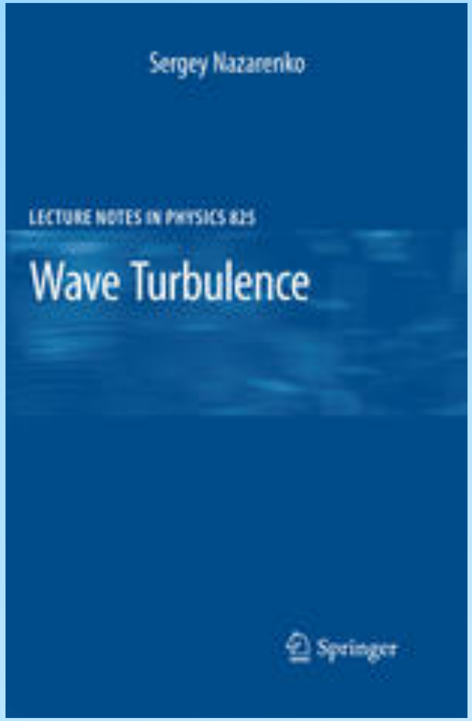
Starting with a helical structure

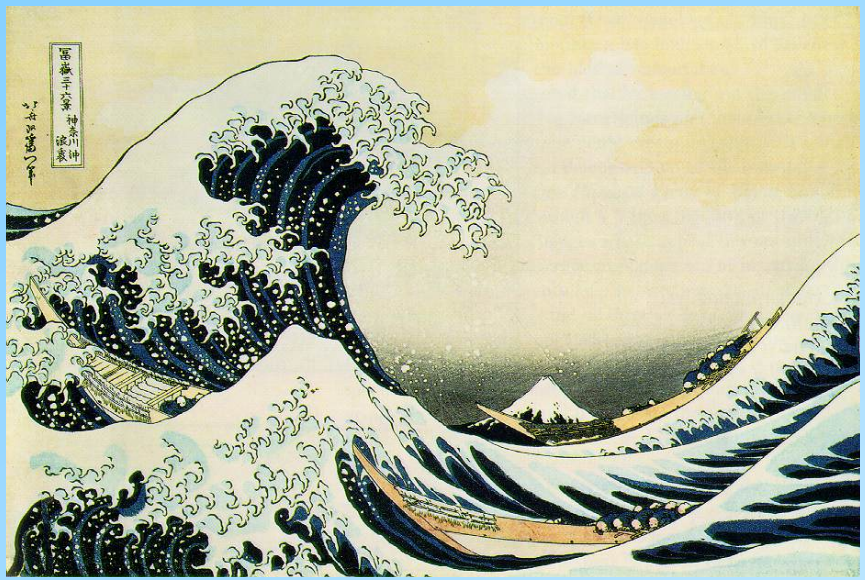
Possible to observe the spin field.

2-4. Bogoliubov wave turbulence in BECs

"Wave turbulence (WT) can be generally defined as out-of-equilibrium statistical mechanics of random nonlinear waves."

No vortices





Great wave by Japanese artist *Hokusai Katsushika*

Examples of WT

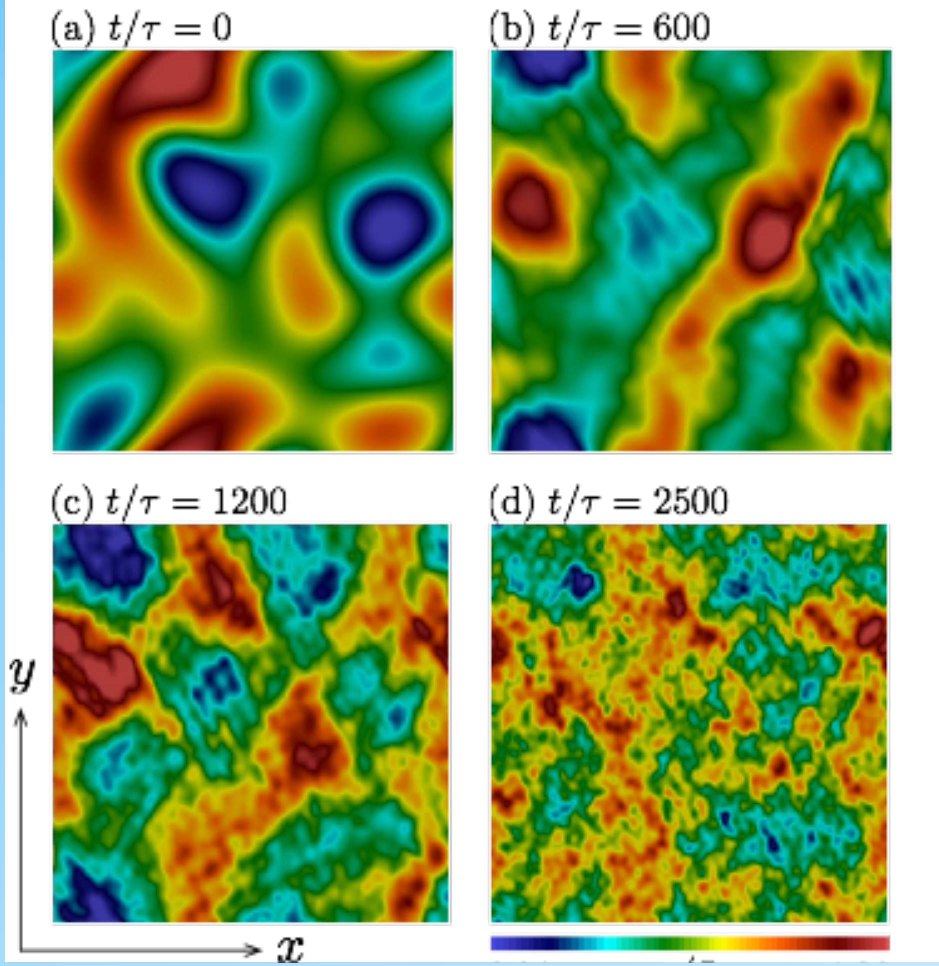
- Acoustic waves
- Surface gravity waves
- Kelvin waves on quantized vortices

etc.

Wave turbulence of Bogoliubov excitations in scalar BEC

K. Fujimoto, M. Tsubota, Phys. Rev. A91, 053620 (2015) ; arXiv:1502.03274

■ Density distribution
(Numerical calculation)



Why is this important in BECs?

・It is very difficult to observe the statistical laws of turbulence on vortices and superflow in cold atoms.

・It can be easier to observe those of density distributions.

Previous study of the wave turbulence by the GP model

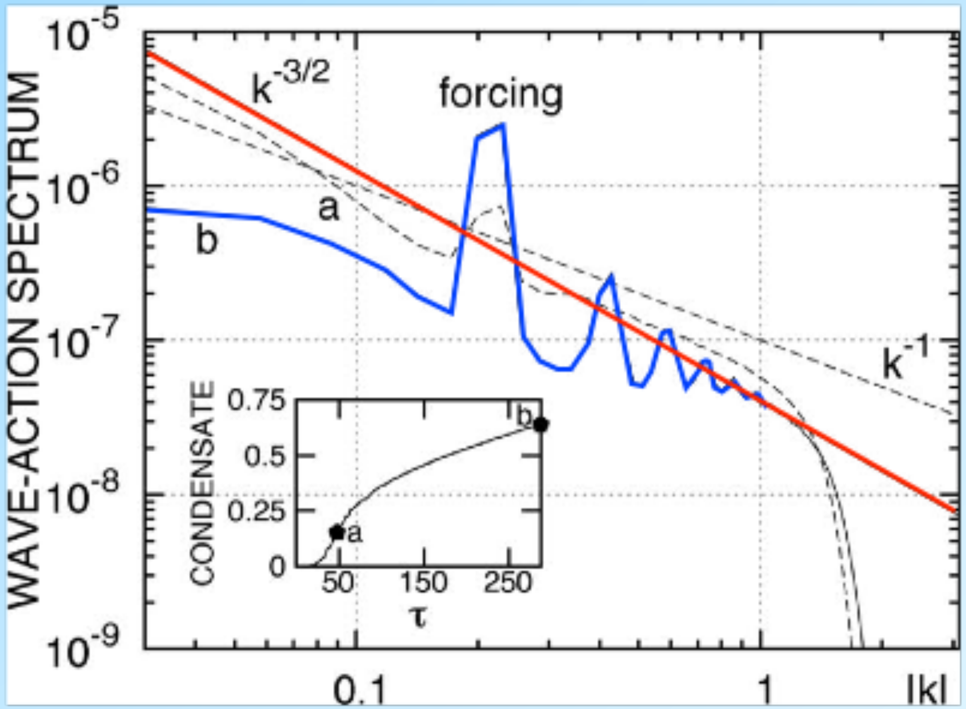
D. Proment, S. Nazarenko, M. Onorato, Phys. Rev. A80, 051603(R) (2009)

Correlation function for wave function

$$C_{\mathrm{w}}(k) = \frac{1}{\Delta k} \sum_{k-\Delta k/2<|\boldsymbol{k}_1|<k+\Delta k/2} \langle|\bar{\psi}(\boldsymbol{k}_1)|^2\rangle$$

The authors derived analytically the power law $C_w \propto k^{-3/2}$ and confirmed it numerically.

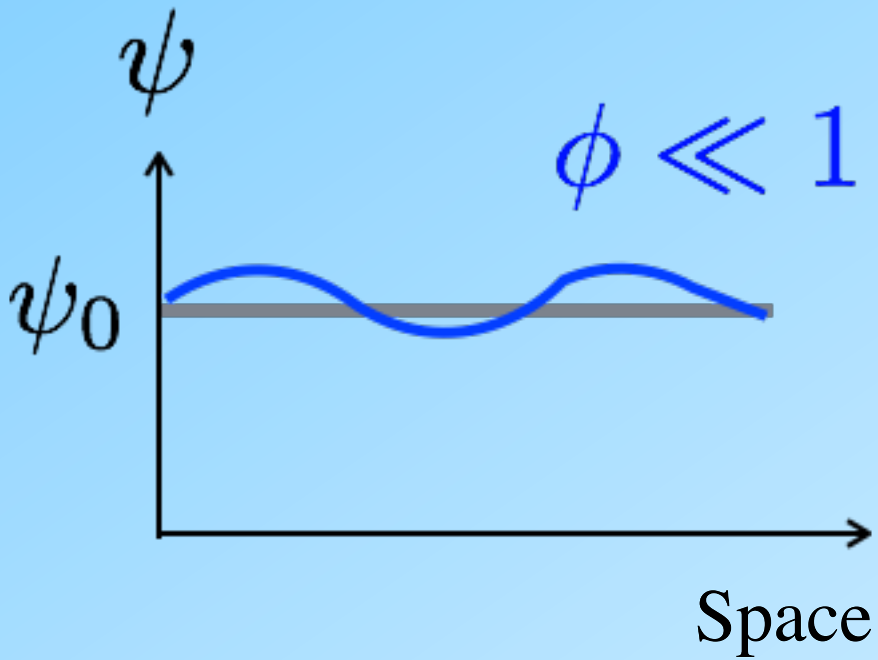
C_{w}



Weak excitation of the GP model

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi$$

1. Uniform system without the trapping potential
2. Weakly interacting Bogoliubov-wave



$$\psi(\boldsymbol{r}) = \psi_0(1 + \phi(\boldsymbol{r}))$$

$$\psi_0 = \frac{1}{L^d}\int \psi(\boldsymbol{r})dV$$

Application of weak wave turbulence (wwt) theory

Substituting $\psi = \psi_0(1+\phi)$ to the GP equation yields

$$i\hbar\frac{\partial}{\partial t}\psi_0 = g\rho_0\psi_0\left[1+\sum_{\boldsymbol{k}_1}(2|\bar{\phi}(\boldsymbol{k}_1)|^2+\bar{\phi}(\boldsymbol{k}_1)\bar{\phi}(-\boldsymbol{k}_1))\right],$$

$$i\hbar\psi_0\frac{\partial}{\partial t}\bar{\phi}(\boldsymbol{k}) = -i\hbar\bar{\phi}(\boldsymbol{k})\frac{\partial}{\partial t}\psi_0+\frac{\hbar^2k^2}{2m}\psi_0\bar{\phi}(\boldsymbol{k})+g\rho_0\psi_0\left[2\bar{\phi}(\boldsymbol{k})+\bar{\phi}^*(-\boldsymbol{k})+2\sum_{\boldsymbol{k}_1\boldsymbol{k}_2}\bar{\phi}^*(\boldsymbol{k}_1)\bar{\phi}(\boldsymbol{k}_2)\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2)\right.$$

$$\left.+\sum_{\boldsymbol{k}_2\boldsymbol{k}_3}\bar{\phi}(\boldsymbol{k}_2)\bar{\phi}(\boldsymbol{k}_3)\delta(\boldsymbol{k}-\boldsymbol{k}_2-\boldsymbol{k}_3)\right],$$

□ Correlation function for wave function

$$C_{\mathrm{w}}(k) = \frac{1}{\Delta k}\sum_{k-\Delta k/2<|\boldsymbol{k}_1|<k+\Delta k/2}\langle|\bar{\psi}(\boldsymbol{k}_1)|^2\rangle$$

□ Correlation function for density distribution

$$C_{\mathrm{d}}(k) = \frac{1}{\Delta k}\sum_{k-\Delta k/2<|\boldsymbol{k}_1|<k+\Delta k/2}\langle|\bar{\rho}(\boldsymbol{k})|^2\rangle$$

WWT theory [1, 2] ⇒

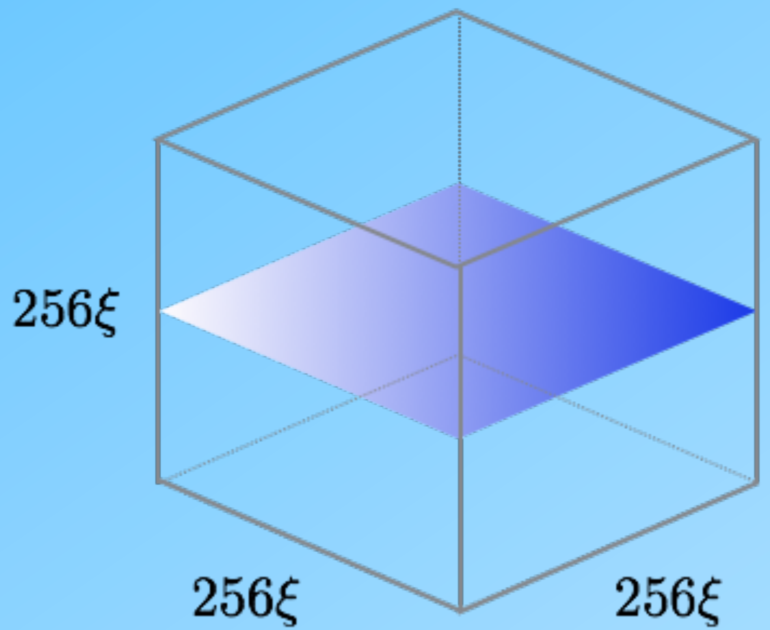
$$C_{\mathrm{w}} \propto k^{-7/2}$$

$$C_{\mathrm{d}} \propto k^{-3/2}$$

[1] V. E. Zakharov et al., Kolmogorov Spectra of Turbulence I: Wave Turbulence.

[2] S. Nazarenko, Wave Turbulence, Lecture Notes in Physics Vol. 825.

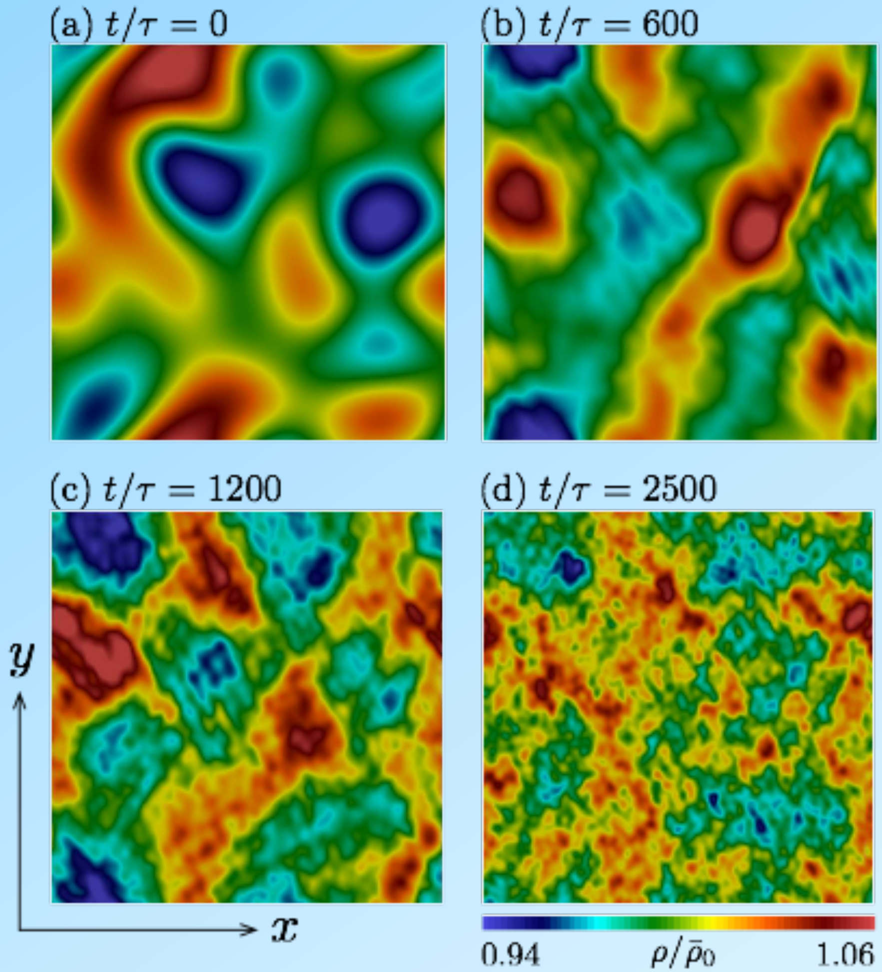
Wave turbulence of the GP model



ξ : Coherence length

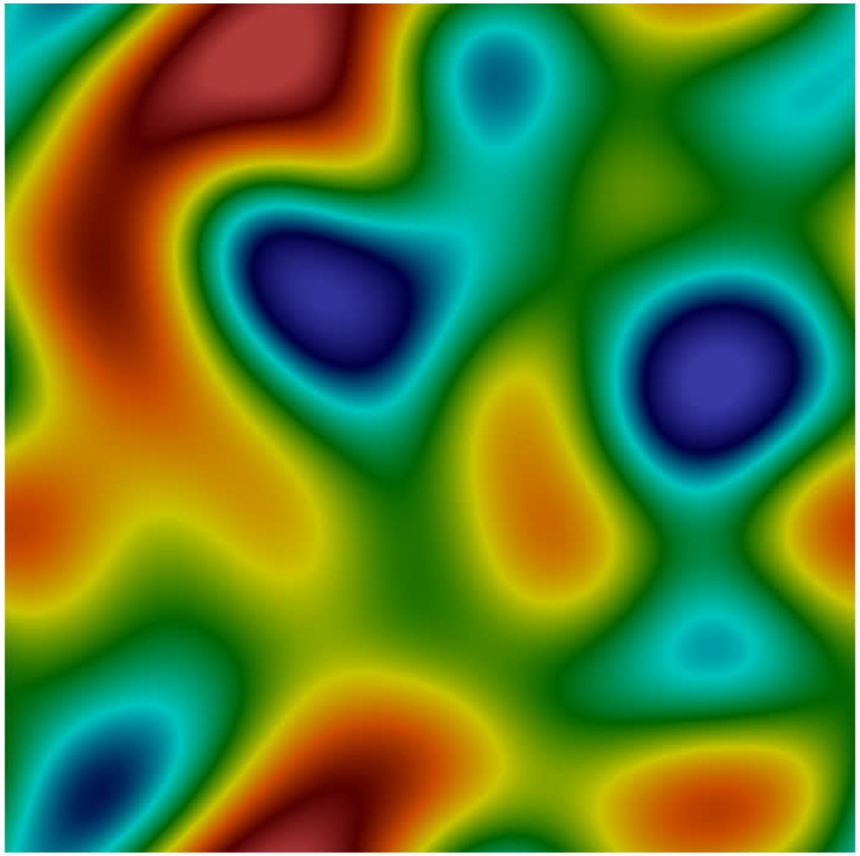
Numerical method : Pseudo-spectral method
Boundary condition : Periodic
Initial state : State generated by random number

◇ Time-development of density distribution

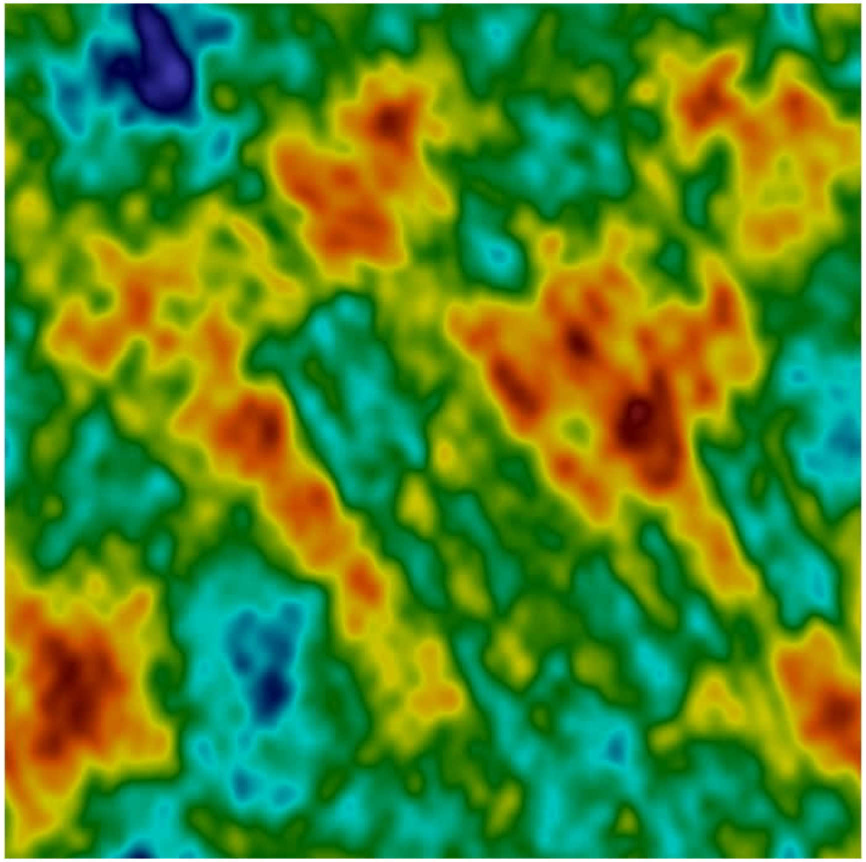


Wave turbulence of the GP model

Early stage (t=0 - 700)



Late stage (t=1500 - 2200)



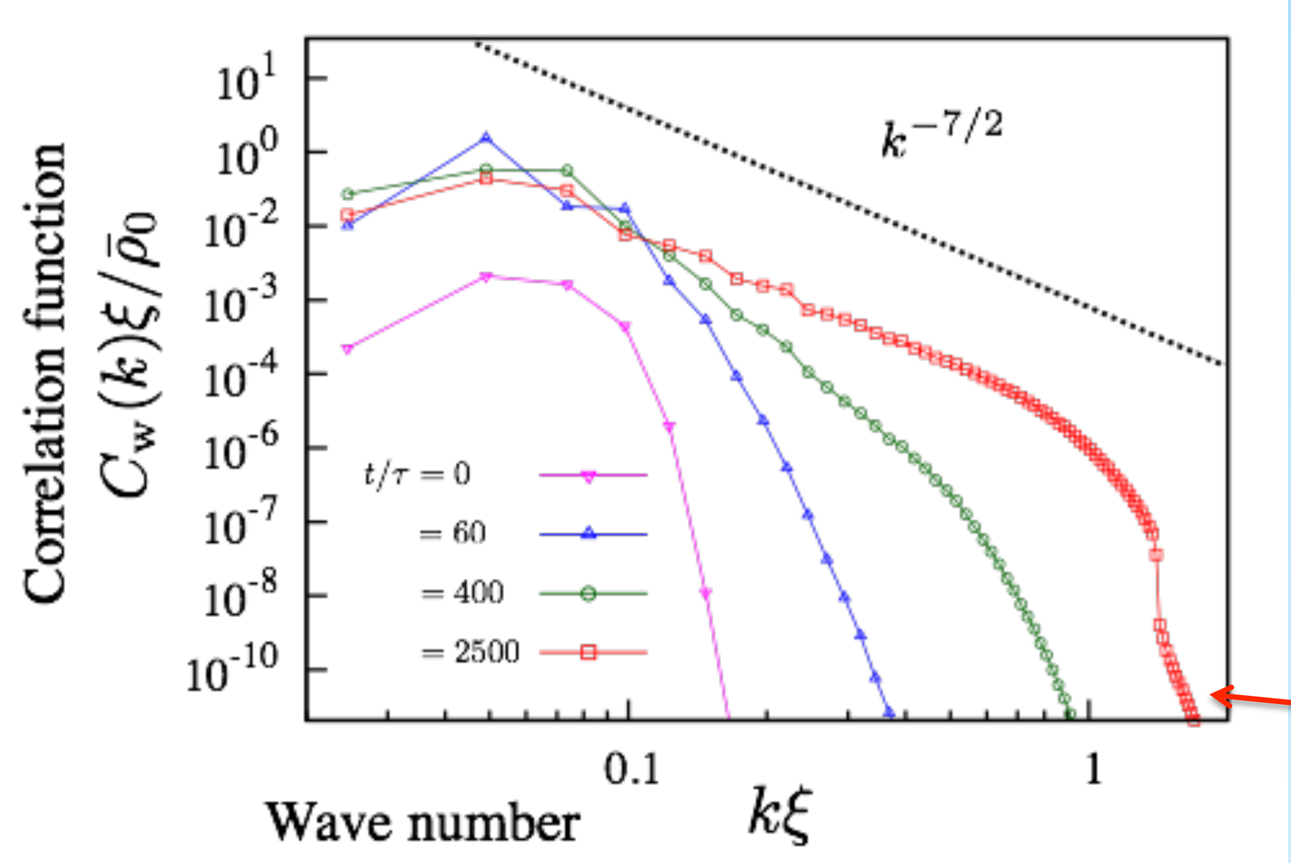
Correlation function of wave function

$$C_w(k) = \frac{1}{\Delta k} \sum_{k-\Delta k/2<|\boldsymbol{k}_1|<k+\Delta k/2} \langle|\bar{\psi}(\boldsymbol{k}_1)|^2\rangle$$

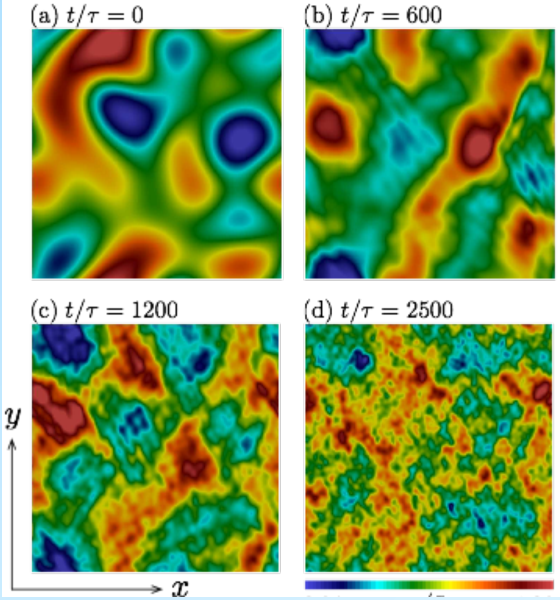
We derive

$$C_w \propto k^{-7/2}$$

Time-development of the correlation function



■ Density distribution
(Numerical calculation)



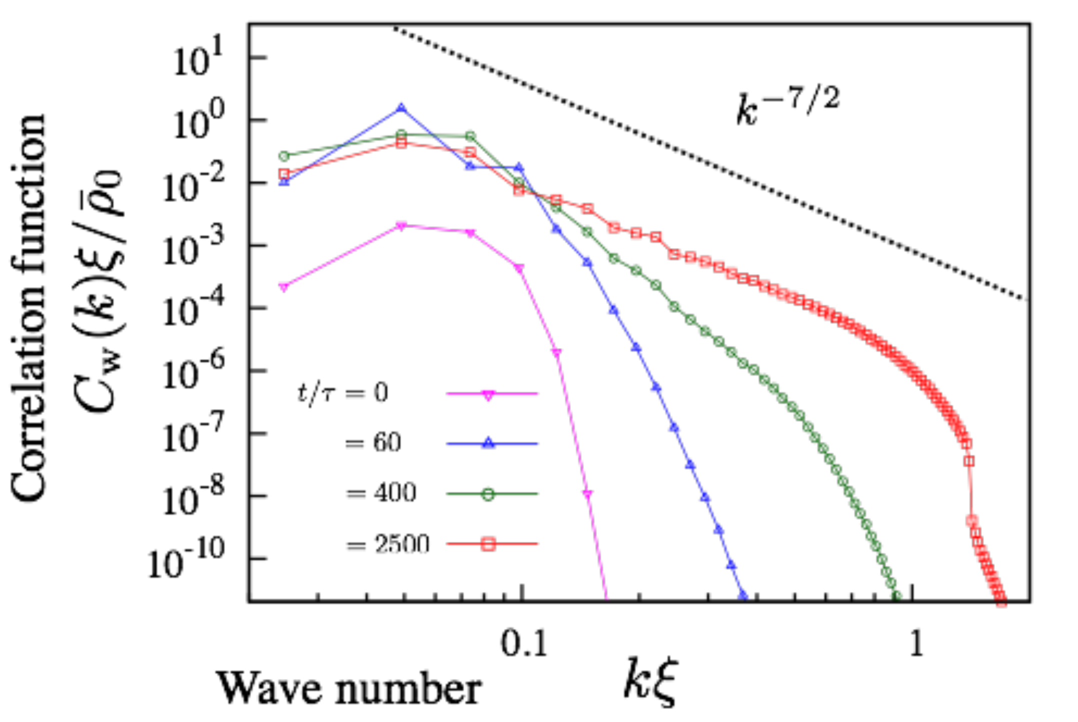
Dissipation is introduced here.

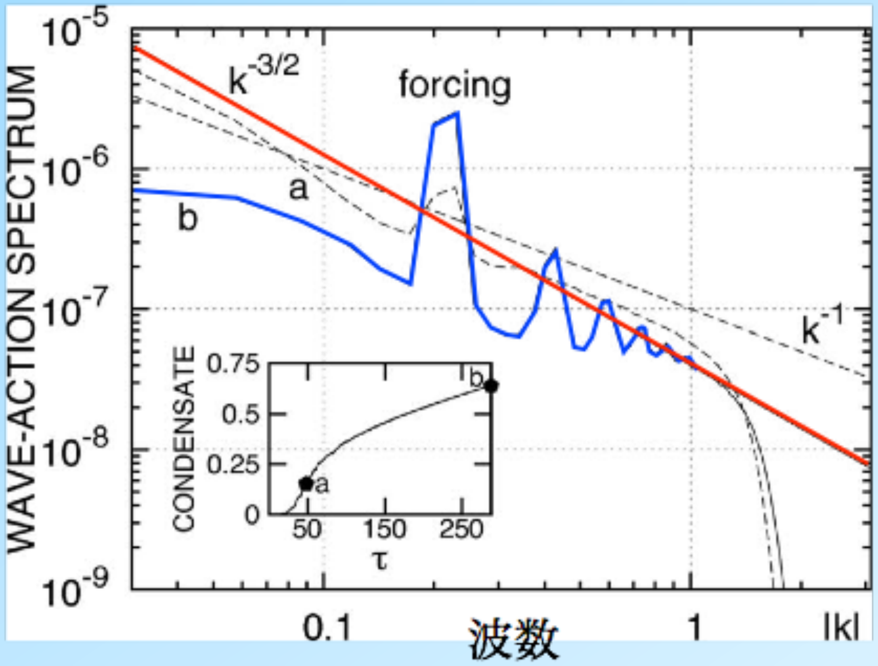
Comparison between ours and the previous one

$$C_{\rm w}(k) = \frac{1}{\Delta k} \sum_{k-\Delta k/2<|\boldsymbol{k}_1|<k+\Delta k/2} \langle|\bar{\psi}(\boldsymbol{k}_1)|^2\rangle$$

Our theory：$C_{\rm w} \propto k^{-7/2}$

Previous study： $C_{\rm w} \propto k^{-3/2}$



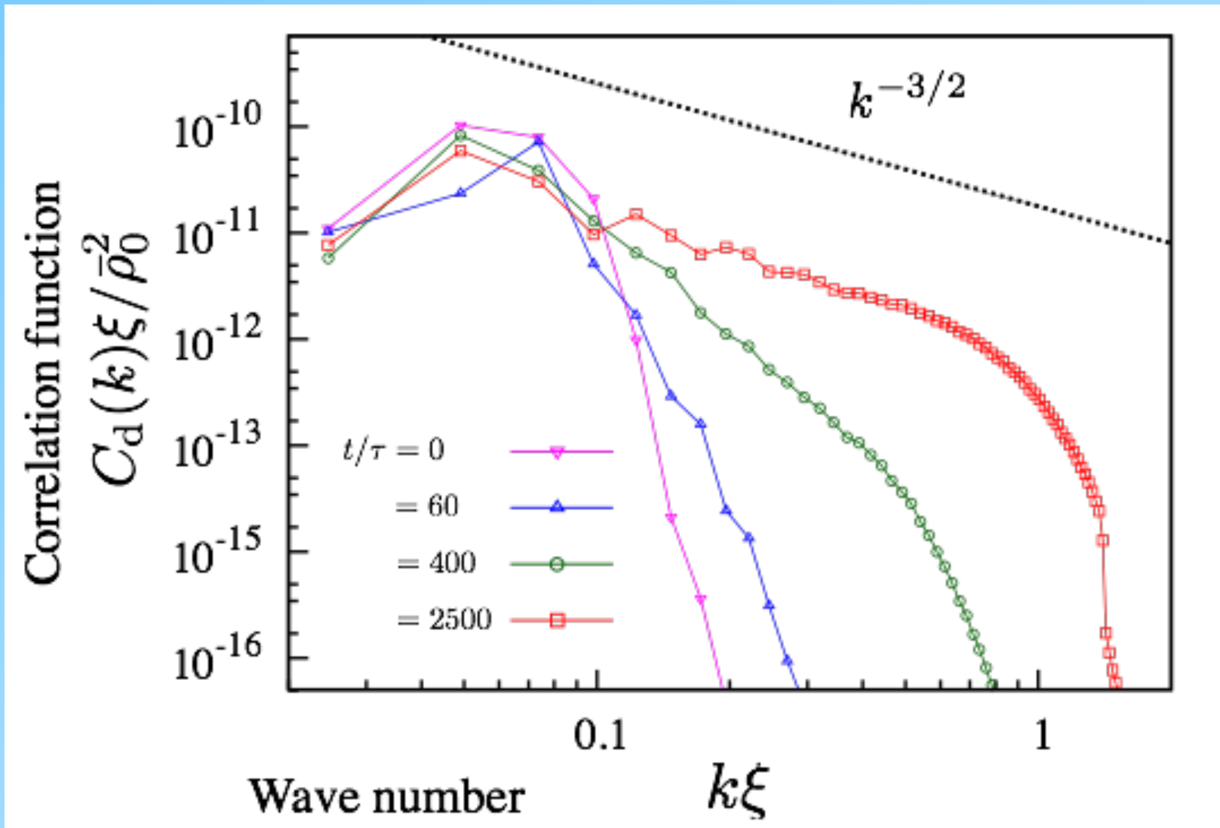


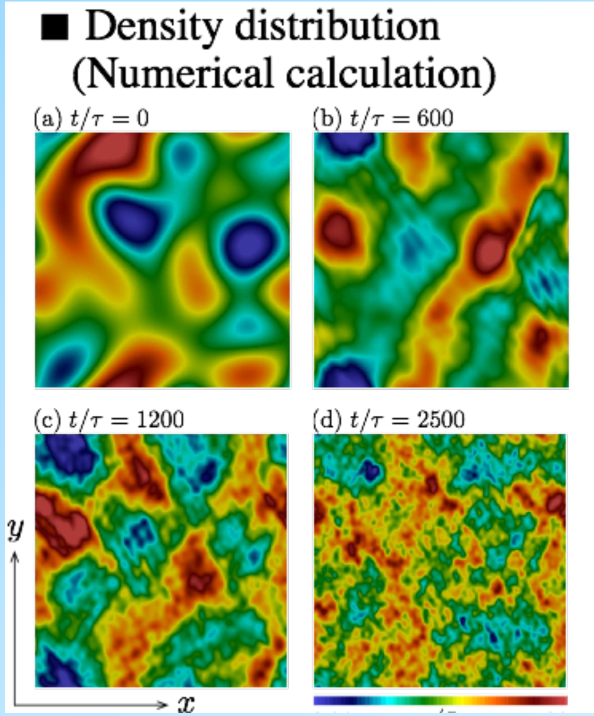
D. Proment et al., Phys. Rev. A 80, 051603(R) (2009)

Correlation function of density distribution

$$C_{\mathrm{d}}(k) = \frac{1}{\triangle k} \sum_{k-\triangle k/2<|\boldsymbol{k}_1|<k+\triangle k/2} \langle |\bar{\rho}(\boldsymbol{k})|^2 \rangle$$

Our theory: $C_d \propto k^{-3/2}$





This density distribution and the cascade can be observed experimentally.

Summary

I discussed three kinds of quantum turbulence in atomic BECs.

Quantized vortices

QT in single-component BECs   QT in two-component BECs

Spins

Spin turbulence in spinor BECs

K. Fujimoto, MT, PRA85, 033642(2012)
MT, Y. Aoki, K. Fujimoto, PRA88, 061601(R) (2013)

Waves

Bogoliubov wave turbulence in BECs

K. Fujimoto, MT, PRA91, 053260(2015)